Ulam - Hyers Stability of Second Order Difference Equations using Laplace Transforms

Malathi S [1], Jothilakshmi R [2]

[1] Department of Mathematics, Muthurangam Government Arts College,

Vellore, India, Affiliated to Thiruvalluvar University; malathi.mga@gmail.com.

[2] Department of Mathematics, Mazharul Uloom College,

Ambur, India, Affiliated to Thiruvalluvar University; jothilakshmiphd@gmail.com.

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Abstract: This article explores the Ulam-Hyers stability of both homogeneous and non-homogeneous second-order difference equations, which are widely applied in control systems and digital signal processing. Using the Laplace transform, we analyze the convergence of these newly defined difference equations. Additionally, we extend our study to the Ulam-Hyers stability of discrete values (difference equations) within an open subinterval by providing a detailed analysis of their properties through Laplace transformation. To enrich our theoretical findings, numerical examples are stated and these examples demonstrates and extend the applications of our theory.

Keywords: Ulam-Hyers stability, homogeneous, non-homogeneous difference equation.

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1. INTRODUCTION

S.M. Ulam introduced the idea of functional equations and their stability during a notable talk [1] in 1940, which generated considerable interest among mathematicians. Ulam's central question was: "Under what conditions can an approximate solution to a functional equation be considered close to an exact solution?" This inquiry led to the analysis and study of the stability constraints of various functional equations under various conditions. D.H. Hyers was the first to provide a rigorous answer to this problem within the context of Banach spaces, demonstrating that approximate solutions can indeed closely approximate exact solutions under specific conditions [2]. This result, now referred to as Hyers-Ulam stability, was later expanded upon by Rassias and others, who applied the concept to a wider range of functional equations [3]. Over time, various researchers have further developed and expanded these stability results to encompass different types of functional equations, employing a wide range of methods and techniques [2]. While the stability of continuous and discrete values has been thoroughly explored in various fields, and the stability of difference equations has received relatively huge attention in control systems and communication engineering.

Popa made a significant contribution by extending the analysis of stability of difference equations by using Hyers-Ulam's stability. He concentrated on Banach spaces and demonstrated their stability in this context [4]. This work underscored the relevance of Ulam's problem to difference equations, effectively substituting the functional equation with a differential equation framework.

Despite these advancements, the exploration of Ulam-Hyers stability in the context of difference equations remains relatively underdeveloped. Further research is needed to explore more general conditions and types of difference equations, particularly in higher-order cases and in different functional spaces. Expanding the understanding of this stability could have significant implications for various applications in control theory, digital signal processing, and beyond.

Consider the difference equation of nth order which is said to exhibit Ulam-Hyers stability in the positive sense if, for any $M > 0$ and any sequence u_k that satisfies the inequality [5]

$$\|F((u_k, u_{k+1}, \ldots, u_{k+n})\| \le M, \qquad M \in \mathrm{N}, \tag{1}$$

the solution v_k is described as

$$\|U_k - V_k\| \leqslant M_k, M \epsilon N, \quad M_k \in \mathbb{N}, \tag{2}$$

Here M_k depends only on M and $lim_{M\to 0} M_k = 0$. If this is true, then M and M_k are extend to arrive the stability.

This suggests that the difference equation is stable, then small perturbations in the input sequence lead to only minor deviations in the solution sequence. If the conditions hold with the constants M and M_k replaced by suitable functions $\varphi(t)$ and $\chi(t)$, respectively and $\lim_{t\to 0} \chi(t) = 0$, then the difference equation is said to possess Ulam-Hyers stability in the generalized sense.

Additionally, consider a related inequality to further explore the stability:

$$|F(u_k, u_{k+1}, \ldots, u_{k+n}) - F(v_k, v_{k+1}, \ldots, v_{k+n})|^{\le} \psi(t), \tag{3}$$

where $\psi(t)$ is a suitable function that diminishes as t approaches zero. This further characterizes the sensitivity of the equation to initial perturbations, providing stability even the system oscillates.

This generalization allows us to apply Ulam-Hyers stability to a broader class of difference equations, thus extending the applicability of the stability concept to more complex systems.

The Z transform serves a crucial role in evaluating in discrete-time systems. This paper explores the Ulam-Hyers stability of both homogeneous and non-homogeneous second-order difference equations by employing the Laplace transform. Typically, Z transform is the method of choice for solving difference equations due to its direct applicability to discrete systems. However, in this study, we utilize the Laplace transform to address these equations, offering a novel approach that leverages of transfer functions. The use of Laplace transform in this context is not only innovative but also provides a more straightforward and effective computational framework, which can be beneficial in solving a wider range of problems across various scientific and engineering disciplines.

By applying the Laplace transform, we can obtain unconditional analytical solutions to difference equations, thus bypassing some of the complexities associated with traditional methods. This approach simplifies the process of finding solutions and extends the utility of the Laplace transform beyond its conventional applications in continuous systems. Moreover, this method enhances the accuracy and efficiency of solving difference equations, making it a valuable tool in fields such as control theory, signal processing, and other areas where discrete systems are prevalent.

This paper explores the adaptation of the Laplace transform for solving difference equations, presenting it as a promising method for both theoretical research and practical applications. By linking continuous and discrete analysis, this approach provides a strong alternative to conventional methods and supports the advancement of mathematical tools in engineering and scientific problem-solving [6].

2. PRELIMINARIES

In this section, we explore the core concepts and key results associated with discrete values of Laplace transform by arriving its stability. In general transform techniques are used in analyzing time-invariant systems. It is applied in control theory, signal processing, and other areas where it simplifies complex problem-solving.

In addition to its utility in solving differential equations, the Laplace transform also aids in stability analysis, system modelling, and the study of transient and steady-state behaviours. This transform's ability to handle initial conditions seamlessly and its application in the study of convolution operations further demonstrate its importance in engineering and mathematical analysis.

2.1. **Properties of Laplace transform.** We outline the key constraints of the Laplace transform, which play a crucial role in its application to various mathematical and engineering problems:

Time-Shifting Property: Time shift of any function results in the following transform:

$$L[f(k - k_0)] = e^{-} k_0^s F(s),$$

where k_0 is a constant. This shifting property is particularly useful in solving differential equations where initial conditions are applied at a time other than zero.

Dirac Delta Function: Mathematically, it is characterized by the property:

$$\delta(t - c) = 0 \quad \text{for} \quad t \neq c,$$

and is typically used in scenarios involving impulse responses. This is particularly significant because it allows for the modeling of systems subject to instantaneous impulses. Additionally, $\delta(t - c) = \infty$ at $t = c$, indicating that the function has an infinite value at this specific point [8].

3. LAPLACE TRANSFORM FOR DIFFERENCE EQUATIONS

Let $f(k) = b[k]$ and $b > 0$, then $f(k)$ is a exponential order and by definition,

$$L[f(k)] = \int_0^\infty e^{-sk} f(k)dk = \int_0^\infty e^{-sk} b(k)dk$$

simplifying, $L[f(k)] = \frac{1-e^{-s}}{k[1-ke^{-s}]}$. We know that $L[U(k)] = U(S)$. Then by using

Heaviside expansion, we get

$$\frac{1-e^{-s}}{s(1-be^{-s})}=b^n, \qquad for \qquad n=0,1,2,...$$

$$\frac{1-e^{-s}}{s(1-be^{-s})^2}=nb^{(n-1)},$$

then,

$$L[U(k+2)]=e^2sU(S)-\frac{e^s}{s}(1-e^{-s})$$

and

$$L[U(k+1)] = e^sL[U(k)] = e^sU(S)$$

We applied the Laplace transform to both homogeneous and non-homogeneous difference equations [10]. By utilizing this technique, we were able to derive convergent solutions for these difference equations. The Laplace transform's effectiveness in simplifying the equations allowed us to address complex problems more efficiently, ensuring that the solutions not only exist but also converge under the given conditions [11].

Example 3.1. Consider the following homogeneous equation,

$$U(k+2) - 18U(K+1) + 81U(K) = 0 \tag{4}$$

we know that $L[f(k)] = f(S)$ then

$$\frac{1-e^{-s}}{s(1-be^{-s})}=b^n, \tag{5}$$

$$\frac{1-e^{-s}}{s(1-be^{-s})^2}=nb^{(n-1)}, for \qquad n=0,1,2,... \tag{6}$$

Apply Laplace transform in (3), we get

$$e^{2s}U(s)-\frac{e^{2s}(1-e^{-s})}{s}-18[e^sU(s)]+81U(s)=0$$

$$U(s)=\frac{1-e^{-s}}{s(1-9e^{-s})^2}$$

Proceeding like this, the corresponding inverse Laplace transform, is

$$U(K)=\frac{1-e^{-s}}{s(1-9e^{-s})^2}=n(9)^{n-1}$$

Example 3.2. Consider the following equation

$$e^{2s}U(s)-\frac{e^{2s}(1-e^{-s})}{s}-5e^sU(s)+6U(s)=0$$

$$U(s)=\frac{e^{2s}(1-e^{-s})}{s(e^{2s}-5e^s+6)}$$

simplifying,

$$U(s) = \frac{3(1-e^{-s})}{s(1-3e^{s})} - \frac{2(1-e^{-s})}{s(1-2e^{s})}$$

Then

$$U(K) = L^{-1}[\frac{3(1-e^{-s})}{s(1-3e^{s})}] - L^{-1}[\frac{2(1-e^{-s})}{s(1-2e^{s})}] = 3^{n+1} - 2^{n+1},$$

Example 3.3. Consider the following equation,

$$e^{2s}U(s) - \frac{e^{2s}}{s}(1-e^{-s}) - 8e^{s}U(s) + 16U(s) = \frac{1}{s^2}$$

$$U(s)(e^{2s} - 8e^{s} + 16) = \frac{1}{s^2} + \frac{e^{2s}(1-e^{-s})}{s}$$

$$L[U(K)] = \frac{e^{-2s}}{e^2}(1 + 2(4e^{-s}) + 3(4^2(e^{-2s}) + ...) + \frac{1-e^{-s}}{s(1-4e^{-s})^2}$$

Using inverse Laplace transform,

$$U(K) = L^{-1}[\frac{e^{-2s}}{e^2}(1 + 2(4e^{-s}) + 3(4^2(e^{-2s}) + ...] + L^{-1}[\frac{1-e^{-s}}{s(1-4e^{-s})^2}]$$

4. ULAM-HYERS STABILITY FOR HOMOGENEOUS DIFFERENCE EQUATIONS

The homogeneous linear difference equation (1) exhibits Ulam-Hyers stability by ensuring that $U(k)$ satisfies the relevant inequality [12]. To arrive the Ulam-Hyers stability, consider the following equation,

$$Z\{U(k+2)\} = z^2U\hat{}(z) - z^2U(-1) - zU\hat{}(z) + zU(-1) + U\hat{}(z),$$

where $U(-1)$ is the initial condition, which simplifies to:

$$Z\{U(k+2)\} = (z^2 - pz + q)U\hat{}(z).$$

Thus, the transformed equation becomes:

$$(z^2 - pz + q)\hat{U}(z) = 0.$$

Let $V\hat{}(z)$ be the Z-transform of the solution $V(k)$ to the homogeneous equation [15]. We need

$$\hat{U}(z) = \hat{V}(z) + \frac{\mathcal{Z}\{f(k)\} - (z^2 - pz + q)\hat{V}(z)}{z^2 - pz + q}.$$

$$\left|\hat{U}(z) - \hat{V}(z)\right| \leq \frac{\epsilon}{|z^2 - pz + q|}.$$

When is sufficiently small and $|z^2 - pz + q|$ is bounded away from zero, we can ensure that:

$\left|\hat{U}(z) - \hat{V}(z)\right| \leq \epsilon.$

Consequently, the difference $|U(k) - V(k)| \leq \epsilon$ for the corresponding time-domain functions. Therefore, the homogeneous difference equation exhibits Ulam-Hyers stability.

Theorem 4.1. For every $M > 0$, let $\varphi(k)$ be any positive and exponentially bounded sequence. For any $p,q \in F$, a positive sequence $U(k)$ satisfying the inequality

$$|U(k+2) + pU(k+1) + qU(k)| \; 6 \; M. \tag{7}$$

Proof: Apply the Laplace transform to $H(k)$, which is exponentially bounded [16] and we get,

$$H(s) = e^{2s}L[U(k)] - \frac{e^{2s}}{s}(1 - e^{-s}) + pe^{s}U(s) + qU(s)$$

$$H(s) = (e^{2s} + pe^{s} + q)U(s) - \frac{e^{2s}}{s}(1 - e^{-s})$$

$$U(s) = \frac{H(s)}{(e^{2s} + pe^{s} + q)} + \frac{e^{2s}(1 - e^{-s})}{s(e^{2s} + pe^{s} + q)}$$

simplifying these,

$$U(s) = H(s)e^{-2s}(1 - qe^{-s})^{-2} + \frac{1 - e^{-s}}{s(1 - qe^{-}s)^{2}}$$

$$\therefore U(s) = H(s)e^{-2s}(1 - qe^{-s})^{-2} + \frac{1 - e^{-s}}{s(1 - qe^{-}s)^{2}}$$

$$U(k) = L^{-1}[H(s)e^{-2s}(1 - qe^{-s})^{-2}] + L^{-1}[\frac{1 - e^{-s}}{s(1 - qe^{-s})^{2}}]$$

$$U(k) = L^{-1}[H(s)e^{-2s}(1 + 2qe^{-s} + 3q^{2}e^{-2s} + \ldots) + nq^{n-1}$$

Then we have,

$$e^{2s}V(s) - \frac{e^{2s}(1 - e^{-s})}{s} + pe^{s}V(s) + qV(s) = 0 \tag{8}$$

$$(e^{2s} + pe^{s} + q)V(s) = \frac{e^{2s}(1 - e^{-s})}{s}$$

$$V(s) = -\frac{e^{2s}(1 - e^{-s})}{s(e^{2s} + pe^{s} + q)}$$

By taking inverse laplace we get,

$$V(k) = nq^{n-1} \tag{9}$$

This leads to the Ulam-Hyers stability [17].

Example 4.2. Let $\varphi(k)$ be a positive sequence and for every $M > 0$, there exists $U(k)$ satisfying the inequality

$$|U(k+2) - 4U(k+1) + 4U(k)| 6 M. \tag{10}$$

Proof: Apply the Laplace transform to $H(k)$, which is exponentially bounded [18] and we get,

$$H(s) = e^{2s}L[U(k)] - \frac{e^{2s}}{s}(1 - e^{-s}) - 4e^{s}U(s) + 4U(s)$$

$$\therefore U(s) = H(s)e^{-2s}(1 - 2e^{-s})^{-2} + \frac{1 - e^{-s}}{s(1 - 2e^{-}s)^2}$$

$$\therefore U(k) = H(k)\phi(k) + n2^{n-1} \tag{11}$$

Let the solution of $V(k)$, is applied in Laplace transform then we get,

$$L(v(k)) = V(s) = -\frac{e^{2s}(1 - e^{-s})}{s(e^{2s} - 4e^{s} + 4)}$$

then applying inverse laplace, we get,

$$V(k) = n2^{n-1} \tag{12}$$

Theorem 4.3. Let $\varphi(k)$ and $f(k)$ be any positive and exponentially bounded sequence. For every $M > 0$, for any $p,q \in F$, then the positive sequence $U(k)$ satisfying the inequality

$$|U(k+2) + pU(k+1) + qU(k) - f(k)| 6 M. \tag{13}$$

Proof: Apply the Laplace transform to $H(k)$, which is exponentially bounded and we get,

$$H(s) = (e^{2s} + pe^{s} + q)U(s) - \frac{e^{2s}}{s}(1 - e^{-s}) - f(s)$$

$$U(s) = \frac{H(s) + f(s)}{(e^{2s} + pe^{s} + q)} + \frac{e^{2s}(1 - e^{-s})}{s(e^{2s} + pe^{s} + q)}$$

simplifying,

$$\therefore U(s) = (H(s) + f(s))e^{-2s}(1 - qe^{-s})^{-2} + \frac{1 - e^{-s}}{s(1 - qe^{-}s)^2}$$

$U(k) = L^{-1}[H(s)e^{-2s}(1 + 2qe^{-s} + f(s)e^{-2s}(1 + 2qe^{-s} + 3q^2e^{-2s} + ...) + nq^{n-1}$ (14) Applying Laplace transform to $V(k)$, we get,

$$e^{2s}V(s) - \frac{e^{2s}(1 - e^{-s})}{s} + pe^{s}V(s) + qV(s) = 0 \tag{15}$$

$$(e^{2s} + pe^{s} + q)V(s) = \frac{e^{2s}(1 - e^{-s})}{s}$$

$$V(s) = -\frac{e^{2s}(1 - e^{-s})}{s(e^{2s} + pe^{s} + q)} = \frac{1 - e^{-s}}{s(1 - qe^{-s})^2}$$

Taking inverse laplace, we get, $V(k) = nq^{n-1}$. Then by definition, the linear non-homogeneous difference equation has Ulam-Hyers stability. We illustrate this theorem 4.4 by the following example. Figures 1 to 6, shows the exact solution and noisy solution for 20 dB to 45dB's.

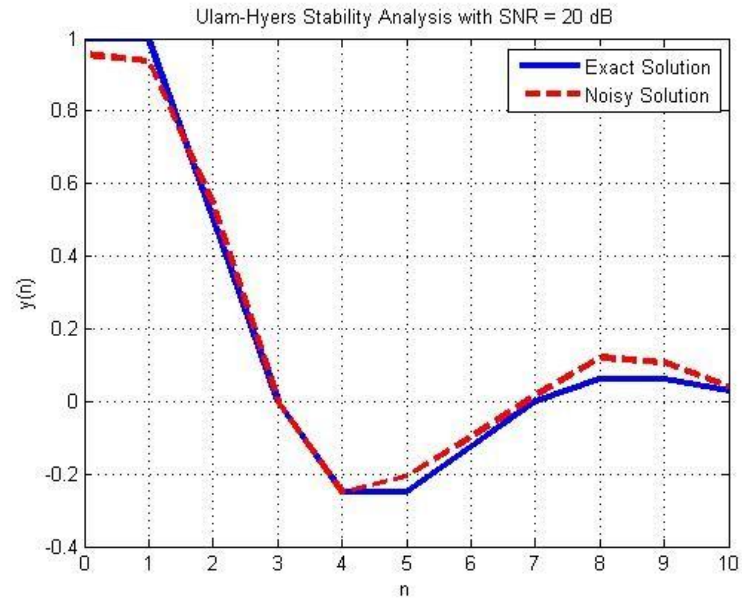

Figure 1. Hyer Ulam Stability for signal to noise ratio for 20 dB

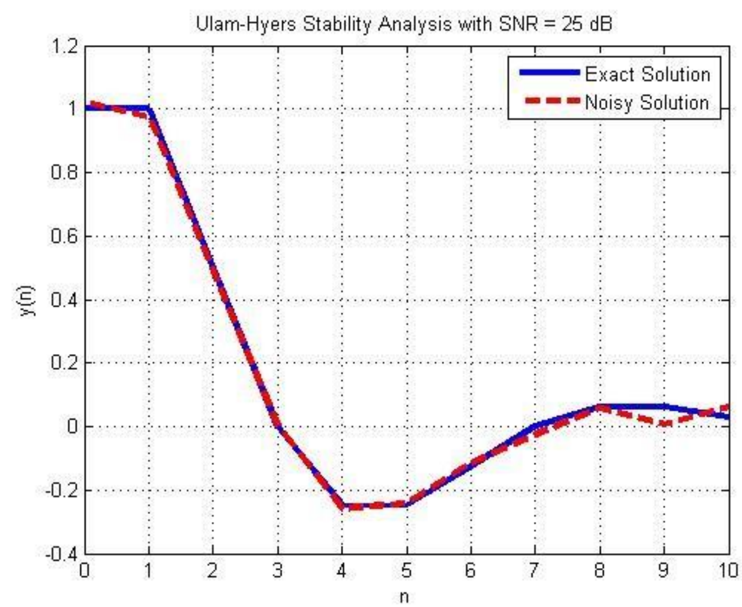

Figure 2. Hyer Ulam Stability for signal to noise ratio for 25 dB

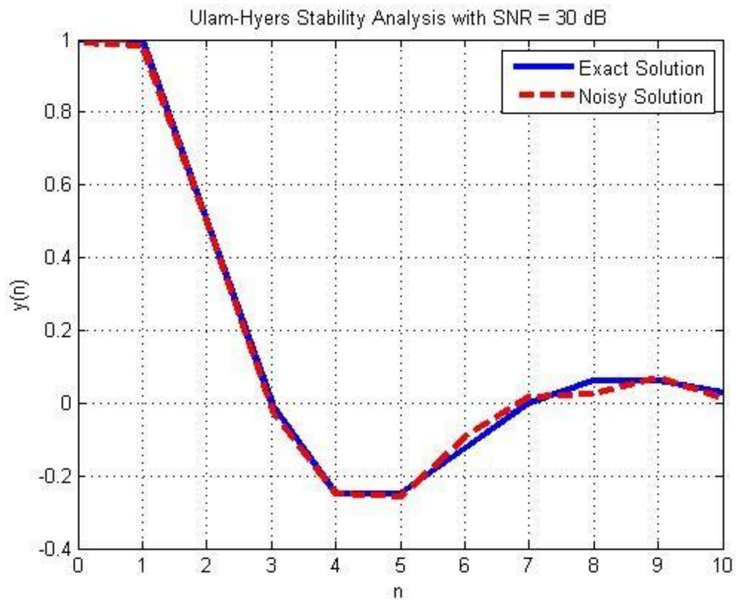

Figure 3. Hyer Ulam Stability for signal to noise ratio for 30 dB

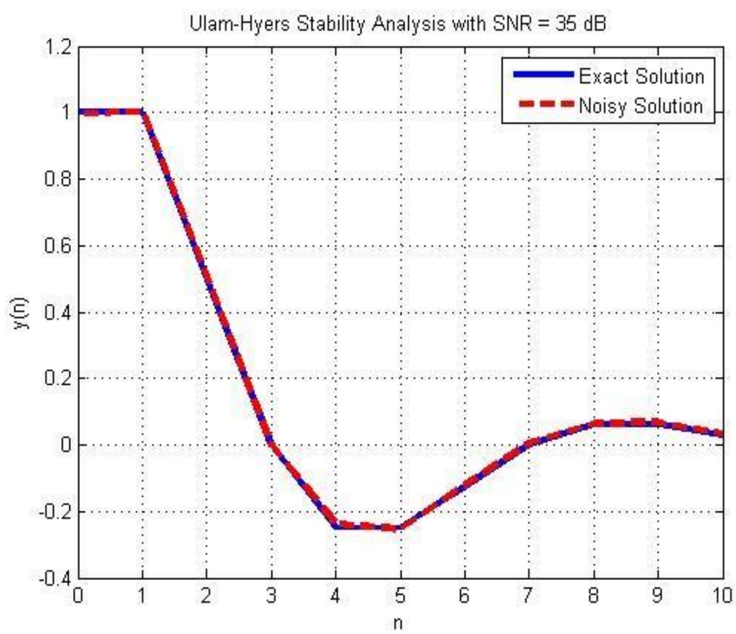

Figure 4. Hyer Ulam Stability for signal to noise ratio for 35 dB

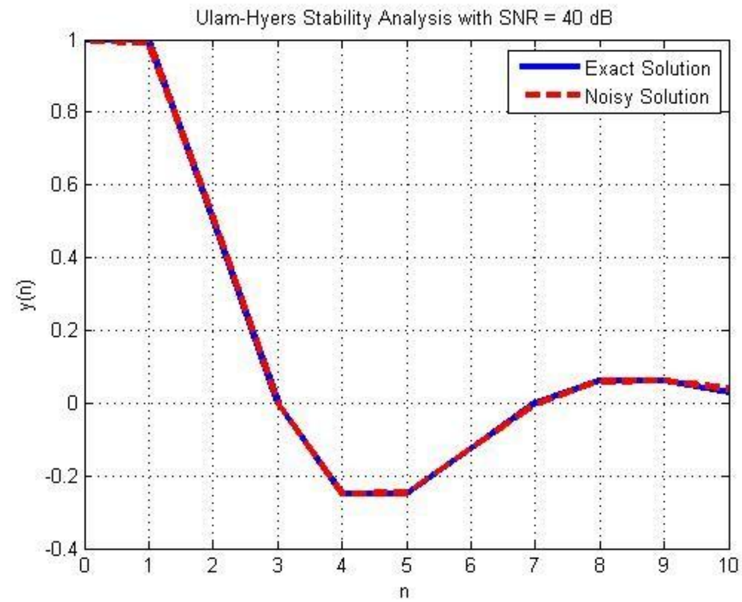

Figure 5. Hyer Ulam Stability for signal to noise ratio for 40 dB

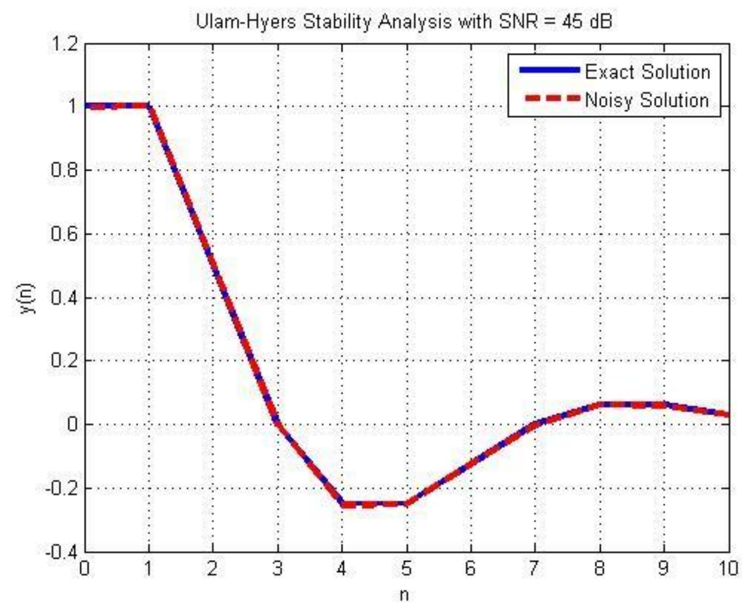

Figure 6. Hyer Ulam Stability for signal to noise ratio for 45 dB *Example* 4.4. Let $\varphi(k)$ be any positive sequence and for every $M > 0$, then the sequence $U(k)$ satisfying the inequality

$$|U(k+2) - 6U(k+1) + 9U(k)| \; 6 \; M, \qquad (16)$$

for all $k \in N$ then the solution $V(k)$ such that $|U(k) - V(k)| \; 6 \; \varphi(k)M$.

Proof: Apply the Laplace transform to $H(k)$, which is exponentially bounded and we get,

$$H(s) = e^{2s}L[U(k)] - \frac{e^{2s}}{s}(1 - e^{-s}) - 4e^{s}U(s) + 4U(s)$$

$$\therefore U(s) = H(s)e^{-2s}(1-2e^{-s})^{-2} + \frac{1-e^{-s}}{s(1-2e^{-s})^2}$$

Substitute $V(k)$, vaalue in the Laplace transform and we get,

$$L(v(k)) = -\frac{e^{2s}(1-e^{-s})}{s(e^{2s}-4e^{s}+4)}$$

$$V(s) = \frac{1-e^{-s}}{s(1-2e^{-s})^2}$$

then applying inverse laplace, we get,

$$V(k) = n2^{n-1} \tag{17}$$

Then by theorem (4.3), the above equation leads to the discrete values of Ulam-Hyers stability.

5. CONCLUSION

In this study, we have established the Ulam-Hyers stability for both second order homogeneous as well as non-homogeneous difference equations. Our results also extend to difference equations with non-constant coefficients, providing a comprehensive method for addressing Ulam-Hyers stability. This enhances the efficiency of solving these difference equations, proving to be a valuable tool in stability analysis. Additionally, this paper shows that the solutions obtained for these difference equations are not only convergent but also exponentially bounded, with well-defined limits. This underscores the reliability and effectiveness of our method in ensuring the stability of difference equations under the given conditions.

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