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Application of Bayesian Techniques and L-Moments I Regional Frequency Analysis of Rainfall Extremes: Insight from Northern Pakistan

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Abstract: Accurate estimation of rainfall extremes is critical for flood risk management, infrastructure design, and climate adaptation particularly in data-scarce, mountainous regions like northern Pakistan, where hydro-meteorological hazards are intensifying due to climate change. This study presents a robust regional frequency analysis (RFA) framework that integrates L-moment-based regionalization with Bayesian hierarchical modeling to improve the estimation of extreme rainfall quantiles. Using annual maximum rainfall data (2006–2023) from seven stations across the Hindu-Kush and Karakoram ranges, the analysis identifies the Generalized Logistic (GLO) distribution as the optimal model, as determined by L-moment ratio diagrams and the ZDIST statistic. The region is confirmed to be acceptably homogeneous ($H < 1$), with no discordant sites ($Di < 1.91$), allowing for the reliable pooling of data. Both L-moment and Bayesian MCMC methods are employed for parameter and quantile estimation, with the Bayesian approach yielding more conservative and uncertainty-aware predictions, particularly for long return periods. At-site quantiles derived from regional growth curves show strong alignment with observed rainfall in 2022–2023, validating the model's predictive accuracy. The Bayesian framework, with its superior handling of small-sample uncertainty and parameter variability, is shown to outperform classical methods, offering a more resilient tool for hydrological risk assessment. These findings underscore the value of integrating Bayesian inference into RFA for climate-resilient water resources planning in vulnerable high-mountain environments.

Keywords: Regional Frequency Analysis (RFA), L-moments, Bayesian Hierarchical Modeling, Generalized Logistic (GLO) Distribution, Extreme Rainfall Quantiles, Markov Chain Monte Carlo (MCMC), Flood Risk Assessment, Northern Pakistan, Hindu-Kush Himalayan Region, At-Site and Regional Quantiles.

INTRODUCTION

The accurate estimation of rainfall extremes is a cornerstone of hydrological design, flood risk management, and climate adaptation planning particularly in mountainous and climatically sensitive regions where extreme precipitation events can trigger catastrophic landslides, flash floods, and infrastructure failure. In recent decades, the increasing frequency and intensity of hydro-meteorological extremes, exacerbated by climate change, have highlighted the limitations of traditional at-site frequency analysis, particularly in data-scarce regions where long-term rainfall records are sparse, unreliable, or nonexistent (IPCC, 2021; Mirza, 2024).

To address these challenges, Regional Frequency Analysis (RFA) has emerged as a robust statistical framework for estimating extreme rainfall quintiles by pooling information across multiple sites within a hydro logically homogeneous region. By transferring knowledge from gauged to uncaused locations, RFA enhances estimation reliability and reduces uncertainty, making it indispensable for water resources planning in developing countries (Hosking & Wallis, 1997; Parajka et al., 2005). A critical component of modern RFA is the use of L-moments**,** a powerful alternative to conventional product moments for characterizing the shape of extreme value distributions. L-moments are more robust to sampling variability and outliers, provide better parameter estimation for skewed and heavy-tailed distributions (e.g., Generalized Extreme Value, GEV), and are less sensitive to data contamination qualities essential for hydrological applications (Viglione, 2014; Serinaldi & Kilsby, 2020). The Index Flood method**,** combined with L-moments and heterogeneity measures (H-statistics), has become the standard approach in regional hydrology (Hosking & Wallis, 1997; Yilmaz & Perera, 2014).

However, conventional RFA methods remain largely deterministic, offering limited quantification of uncertainty in parameter estimates and return level predictions. This limitation is particularly problematic in risk-sensitive decision-making, where confidence bounds and probabilistic forecasts are essential for resilient infrastructure design and disaster preparedness.

To overcome this, Bayesian hierarchical modeling has been increasingly integrated into RFA frameworks. Unlike classical frequents approaches, Bayesian methods explicitly model uncertainty at multiple levels data, parameters, and model structure and allow for the incorporation of prior information, such as physical constraints, regional climate patterns, or expert judgment (Renard, 2011; Sun et al., 2022). This is especially valuable in regions like northern Pakistan, where observational networks are sparse but prior knowledge about monsoon dynamics and topographic influences on rainfall can be leveraged.

Recent studies have demonstrated the superiority of Bayesian RFA in improving predictive accuracy and uncertainty quantification. For instance, Brunner et al. (2019) showed that Bayesian pooling strength across sites yields more reliable estimates of high quantiles, while Huang et al. (2021) applied Bayesian spatial models to account for geographical covariates in regional rainfall frequency. In the South Asian context, Mirza (2024) emphasized the urgent need for advanced statistical tools to assess changing precipitation extremes under climate variability, particularly in the Hindu-Kush Himalayan region, which is warming at nearly twice the global average rate. The northern areas of Pakistan encompassing Gilgit-Baltistan and Khyber Pakhtunkhwa represent one of the most hydro logically complex and vulnerable regions in the world. Characterized by extreme topography, glacial systems, and intense rainfall driven by monsoons and western disturbances, this region is highly susceptible to flash floods and debris flows. However, rainfall monitoring remains inadequate, with many stations exhibiting short or discontinuous records (Khan et al., 2023). Conventional frequency analysis in such settings is fraught with uncertainty, and the impacts of climate change further complicate stationarity assumptions.

To address these shortcomings, recent studies have advocated for the integration of Bayesian hierarchical models into RFA. Bayesian methods offer a robust framework for quantifying uncertainty, allowing for the explicit incorporation of prior knowledge, spatial covariates, and model uncertainty (Renard, 2011; Sun et al., 2022). For instance, Brunner et al. (2019) demonstrated that Bayesian pooling improves the reliability of extreme quantile estimates, while Huang et al. (2021) showed that spatial Bayesian models outperform classical RFA in regions with complex climatology.

Despite these advances, a significant gap remains in the literature regarding the application of Bayesian RFA in high-mountain, data-scarce environments, such as northern Pakistan. Existing studies in South Asia have primarily focused on large river basins or monsoon-dominated regions, with limited attention to orographic rainfall extremes and non-stationary climate **impacts** in the western Himalayas (Rahman et al., 2020). Furthermore, most regional analyses in Pakistan have not incorporated topographic or climatic covariates (e.g., elevation, slope, and distance from moisture sources) into the modeling framework, despite their known influence on rainfall distribution. By doing so, this research responds to the urgent need for climate-resilient hydrological modeling in one of the world's most hazard-prone regions. The findings will support policymakers, engineers, and environmental planners in developing adaptive strategies for flood risk mitigation in the face of growing climate uncertainty.

OBJECTIVE OF THE STUDY

The primary objectives of this research are as follows.

- To evaluate the suitability and homogeneity of rainfall data from gauged sites in seven regions of Pakistan for Regional Frequency Analysis (RFA).
- To estimate regional rainfall quantiles for extreme events in the northern areas of Pakistan.
- To derive at-site quantile estimates using regional growth curves for ungauged or data-scarce locations.
- To compare the performance of Bayesian estimation and L-moment-based methods in quantile estimation.
- To validate the RFA model by comparing predictions from the 2006–2023 AMRS with observed 2018 rainfall data.

LITERATURE REVIEW

Extreme environmental events such as floods, droughts, and intense rainfall episodes have become increasingly frequent and severe in recent decades, posing significant threats to human societies, infrastructure, and economic stability. The accurate prediction and modeling of such events are critical for climate adaptation, disaster risk reduction, and sustainable water resource management. In this context, extreme value theory (EVT) and regional frequency analysis (RFA) have emerged as essential tools for estimating high-quantile events, particularly in regions where observational records are short or spatially sparse. The foundation of modern extreme value analysis was laid by Hosking et al. (1985), who introduced Probability Weighted Moments (PWM) for parameter estimation of the Generalized Extreme Value (GEV) distribution. Their work demonstrated that PWM estimators exhibit smaller bias and greater efficiency, especially in small samples, compared to classical Maximum Likelihood Estimation (MLE). This seminal contribution paved the way for more robust and reliable modeling of extreme hydrological events.

Building on this, Hosking (1990) introduced L-moments as a superior alternative to conventional product moments. L-moments are based on linear combinations of order statistics, making them more robust to outliers, less sensitive to sampling variability, and more effective for small-sample inference. These properties have made L-moments the standard method in hydrological frequency analysis, particularly within the framework of the Index Flood method (Hosking & Wallis, 1997).

The integration of Bayesian methods into extreme value modeling marked a significant advancement in uncertainty quantification. Coles and Tawn (1996) applied Bayesian inference to 54 years of daily rainfall data from southwest England, using informative priors elicited from expert judgment. Their results showed that the Bayesian approach provided more realistic credible intervals and outperformed MLE in small-sample settings. This study highlights the importance of incorporating prior knowledge into hydrological models, particularly in situations of data scarcity. Further developments in regional modeling were advanced by Madsen et al. (1997), who compared the Partial Duration Series (PDS) and Annual Maximum Series (AMS) approaches using data from 48 catchments in New Zealand. They found that PDS models, based on the Generalized Pareto (GPA) distribution, yielded more homogeneous regional groupings than AMS models, which assume the GEV distribution. This highlighted the importance of data selection in RFA.

In North America, Katz et al. (2002) analyzed a 10-year AM series of daily precipitation in Fort Collins, Colorado, using both MLE and Bayesian techniques. They demonstrated that the Bayesian framework not only improved parameter estimation but also provided a coherent assessment of uncertainty, making it particularly valuable for infrastructure design under climate uncertainty. The use of Markov Chain Monte Carlo (MCMC) methods in Bayesian inference was popularized by Hitchcock (2003), who explained the Metropolis-Hastings (M-H) algorithm as a powerful tool for simulating from complex posterior distributions. This computational advance enabled the practical application of Bayesian models to real-world hydrological problems.

Ries and Stedinger (2005) compared Bayesian MCMC, MLE, and PWM for estimating flood quantiles using lognormal and Log-Pearson Type III (LP-III) distributions. They concluded that Bayesian MCMC offered superior performance in characterizing parameter uncertainty and was more computationally stable than classical methods. In regional studies, Modarres (2006) applied L-moment-based RFA to rainfall data from 28 cities in Iran. Using Z-statistics and homogeneity measures (H), he identified the three-parameter lognormal (LN-3P) as the best-fitting regional distribution, demonstrating the importance of regional homogeneity in model selection. Nadarajah and Choi (2007) modeled 41 years of annual maximum rainfall data from five locations in South Korea using the GEV distribution. Their MLE-based analysis revealed significant spatial variability in return levels, emphasizing the need for region-specific modeling.

Mohssen (2009) compared GPA and GEV distributions for partial duration and annual maximum series in New Zealand, finding that GPA fit PDS better, while GEV was superior for AMS—a result consistent with theoretical expectations. In Asia, Liang et al. (2011) applied Bayesian MCMC to 33 years of peak flow data from Pingyuan, China, estimating quantiles of the Pearson Type III (P-III) distribution. Their use of uniform priors and MCMC sampling demonstrated the feasibility of Bayesian methods in data-limited regions. Eli et al. (2012) and Isikwue et al. (2015) independently applied Bayesian MCMC and MLE to rainfall data in Malaysia and Nigeria, respectively. Both studies found that Bayesian methods outperformed MLE in small-sample settings, with lower Percent Bias (PBIAS), RMSE, and MAE, particularly when using non-informative priors.

Shaizadi et al. (2013) and Ahmad et al. (2013, 2016) conducted RFA in Pakistan using L-moments and Kappa/GEV distributions. Ahmad et al. (2016) analyzed 28 sites across Pakistan and found GEV to be the best-fit distribution at most locations, based on RMSE and ratio plots. Their work highlighted the spatial heterogeneity of rainfall extremes in the country. Ghosh et al. (2016) evaluated six distributions for monthly rainfall data in Bangladesh. They found GEV to be the best fit for three out of four stations, reinforcing its applicability in South Asia. Alahamadi (2017) applied RFA to rainfall data in Medina, Saudi Arabia, identifying Pearson Type III (PE3) as the optimal regional distribution using L-moment-based goodness-of-fit tests. In Pakistan, Hussain et al. (2017) used RFA on seven sites in Sindh province, identifying PE3, GNO, and GPA as suitable regional distributions. They also developed regression models between rainfall quantiles and elevation for ungauged sites, enhancing practical utility. Yasmeen et al. (2018) emphasized the use of L-moments in cluster and linkage analysis, advocating for its role in regionalization and rainfall prediction. Ullah et al. (2019) highlighted the increasing frequency of extreme rainfall in northern Pakistan, particularly in agricultural zones. They stressed the importance of L-moment-based RFA for long-term climate resilience. Recent studies have focused on non-stationarity, climate change impacts, and Bayesian hierarchical modeling. Fatima et al. (2022) assessed non-stationary frequency analysis of rainfall extremes in Pakistan, finding significant trends in intensity and frequency.

Khan et al. (2023) applied Bayesian RFA to northern Pakistan, incorporating topographic covariates and demonstrating improved predictive accuracy. Mirza (2024) emphasized the need for adaptive RFA frameworks in response to climate change, particularly in the Hindu-Kush Himalayan region. The evolution of extreme value modeling has transitioned from classical moment-based methods to robust L-moment and Bayesian approaches, with increasing emphasis on regional homogeneity, uncertainty quantification, and climate non-stationarity. The GEV distribution, combined with L-moments and Bayesian MCMC, has become the gold standard in hydrological frequency analysis. In Pakistan, recent studies have validated the effectiveness of these methods, particularly in the data-scarce northern regions. However, challenges remain in accounting for measurement error, non-stationarity, and integrating spatial covariates, which this study seeks to address.

METHODOLOGY:

This study employs a Bayesian regional frequency analysis (RFA) framework to estimate extreme rainfall quantiles in the northern areas of Pakistan a hydro logically complex and climatically sensitive region prone to flash floods, glacial lake outburst floods (GLOFs), and droughts. The methodology integrates L-moment-based regionalization with Bayesian hierarchical modeling to enhance estimation accuracy, explicitly quantify uncertainty, and improve predictive reliability in data-scarce environments.

DATA COLLECTION AND PREPROCESSING

Annual Maximum Rainfall Series (AMRS) data from **seven** meteorological stations: Astore, Bunji, Chillas, Gilgit, Gupis, Hunza, and Skardu were obtained from the Pakistan Meteorological Department (PMD) for the period 2006–2023 (18 years). These stations are located in the Hindu-Kush, Karakoram, and western Himalayan ranges, a region characterized by extreme topography, orographic rainfall, and high climatic variability.

Prior to analysis, the data underwent rigorous quality control, including checks for missing values, outliers, and measurement inconsistencies. Missing data were imputed using regression-based methods with elevation and distance from moisture sources as covariates. The homogeneity of time series was assessed using the Mann-Whitney U test (Mann & Whitney, 1947) and Spearman's rank correlation test for trend detection (Spearman, 1904), ensuring that the series met the assumptions of independence, stationarity, and randomness (Naghettini & Pinto, 2016).

REGIONAL FREQUENCY ANALYSIS FRAMEWORK

The RFA methodology is based on the Index Flood approach (Dalrymple, 1960; Hosking & Wallis, 1997), which assumes that rainfall extremes across a region follow a standard distribution, differing only in scale. The process involves pooling data from multiple sites to enhance the reliability of estimation at both gauged and uncaused locations.

DISCORDANCY AND HETEROGENEITY TESTS

To identify sites with atypical behavior, the discordancy measure (Di) based on L-moments was applied (Hosking & Wallis, 1997). The statistic is defined as:

$$D_i = Di \ = \frac{1}{3} 1/N(\mathrm{u}_i - \bar{\mathrm{u}})^T \mathrm{A}^{-1}(\mathrm{u}_i - \bar{\mathrm{u}})$$

Where u_i is the vector of L-moment ratios (L-CV, L-skewness, L-kurtosis) for site *i*, $\bar{\mathrm{u}}$ is the regional average, and A is the covariance matrix. A site is considered discordant if $D_i > 3$, indicating it should be excluded from the region.

The heterogeneity measure (H) was used to assess regional homogeneity. This statistic compares the observed dispersion of L-moment ratios with that expected from simulated homogeneous regions using the four-parameter Kappa distribution. The region is classified as:

- Acceptably homogeneous if$\leq H<1$
- Possibly heterogeneous if $1 \leq H < 2$
- Definitely heterogeneous if $H \geq 2$

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SELECTION OF BEST-FIT DISTRIBUTION

Candidate distributions Generalized Extreme Value (GEV)**,** Generalized Logistic (GLO)**,** Generalized Normal (GNO)**,** Pearson Type III (PE3)**,** and Generalized Pareto (GPA) were evaluated using L-moment ratio diagrams (LMRDs) and the Z-statistic (Hosking & Wallis, 1997):

$$ZDIST = \frac{T4DIST \ -t4R}{\sigma 4t \ \ +B4}$$

Where τ_4^{DIST} is the theoretical L-kurtosis of the candidate distribution, t_4^R is the regional average L-kurtosis, σ_{t4} is its standard deviation from simulations, and B4 is the bias. A distribution is considered a good fit if $|ZDIST| < 1.64$, corresponding to the 90% significance level.

Bayesian Hierarchical Modeling

To overcome the limitations of classical RFA particularly its inability to quantify parameter uncertainty this study adopts a Bayesian hierarchical model (Renard, 2011; Sun et al., 2022). The model treats distribution parameters as random variables with prior distributions informed by regional data.

The posterior distribution of parameters θ is derived using Bayes' Theorem**:**

$$p\left(\frac{\theta}{\mathbf{y}}\right) \propto p\left(\frac{\mathbf{y}}{\theta}\right) \cdot p(\theta)$$

Where **y** is the observed AMRS data, $p(\mathbf{y}|\theta)$ is the likelihood function (e.g., GEV), and $p(\theta)$ is the prior. Non-informative (flat) priors are used for scale and shape parameters, while the location parameter is modeled with a normal prior centered on the regional mean.

Posterior inference is performed using Markov Chain Monte Carlo (MCMC) sampling via the Metropolis-Hastings algorithm (Hitchcock, 2003), implemented in R using the BayFusion and rFSA packages. Convergence is assessed using Gelman-Rubin diagnostics and trace plots.

QUANTILE ESTIMATION AND UNCERTAINTY QUANTIFICATION

Rainfall quantiles for return periods of 10, 25, 50, and 100 years are estimated from the posterior predictive distribution. For the GEV distribution, the quantile function is:

$$x_T = \mu - \frac{\sigma}{\xi}\left[1-\left(-\ln\left(1-\frac{1}{T}\right)\right)^{-\xi}\right]$$

Where T is the return period, credible intervals (90% and 95%) are computed from the MCMC output, providing a robust measure of uncertainty.

MODEL VALIDATION AND PERFORMANCE ASSESSMENT

The model is validated through:

- Split-sample testing: Comparing predictions from the 2006–2023AMRS with observed 2018 rainfall data.
- Cross-validation: Leave-one-out validation across stations.
- Goodness-of-fit tests: Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D) tests on the fitted distribution.

Performance is evaluated using the Relative Root Mean Square Error (RRMSE) and Nash-Sutcliffe Efficiency (NSE).

ANALYSIS

This section presents a regional frequency analysis of maximum monthly rainfall in northern Pakistan using L-moments and Bayesian methods. Rainfall data from seven stations Astore, Bunji, Chillas, Gilgit, Gupis, Hunza, and Skardu were obtained from the Pakistan Meteorological Department for the period 2006–2023 (18 years). The data, measured in millimeters (mm), show a minimum annual maximum rainfall of 7.18 mm (Bunji) and a maximum of 76.10 mm (Gupis), with average values ranging from 20.95 mm to 42.77 mm across the region.

Table 1: Summary Statistic for Annual Maximum Monthly Rainfall Totals (in millimeter) of Seven Sites of Northern Areas, Pakistan

Name	N	Min	1^{st}Qua	Median	Mean	3^{rd}Qua	Max
Astore	18	19.80	31.23	42.80	43.77	52.12	68.50
Bunji	18	7.80	16.62	25.15	27.32	33.38	56.20
Chillas	18	11.20	22.75	25.85	28.75	34.67	68.00
Gilgit	18	11.00	16.93	20.10	23.62	31.23	64.50
Gupis	16	12.00	15.38	24.00	26.92	34.25	76.10
Hunza	9	9.00	14.40	19.00	20.95	21.35	61.00
Skardu	17	9.80	18.50	29.80	29.76	37.60	64.20

The summary statistics of annual maximum monthly rainfall (in mm) from 2006–2023across seven sites in northern Pakistan reveal substantial spatial variability in rainfall patterns. Astore recorded the highest mean rainfall (43.77 mm) and median (42.80 mm), indicating consistently intense monthly rainfall events. In contrast, Gupis experienced the highest single-year maximum of 76.10 mm, highlighting its susceptibility to extreme precipitation. In contrast, Bunji had the lowest minimum value (7.80 mm), and Hunza had the lowest mean (20.95 mm), suggesting relatively drier conditions in these areas. Gilgit and Hunza also exhibited lower median values (20.10 mm and 19.00 mm, respectively), reflecting less intense rainfall overall. High upper-quartile values in Skardu (37.60 mm) and Chillas (34.67 mm) suggest frequent high-intensity rainfall events. However, data completeness varies, with Hunza having only 9 years and Gupis 16 years of records, potentially affecting the representativeness of their statistics. Overall, Astore and Gupis emerge as hotspots for extreme rainfall, while Gilgit and Hunza experience milder extremes. This pronounced spatial heterogeneity underscores the complexity of hydrological processes in the region. It highlights the importance of site-specific analysis for accurate flood risk assessment and sustainable water resource management in northern Pakistan.

THE ASSUMPTION OF RANDOMNESS

The results of the NERC test, presented in Table 2, indicate that the observed annual maximum rainfall series for each station is random. The test statistic values are relatively low, and the corresponding p-values are greater than the significance level ($p > 0.05$), leading to the acceptance of the null hypothesis of randomness. This suggests that the data points in the series are independently distributed and not influenced by any systematic patterns or dependencies, confirming that the time series meets the assumption of randomness required for reliable frequency analysis.

Table 2: values of NERC, Wald-Wolfowitz, Mann-Whitney, and Spearman Rank Correlation Test at each site of Study are

Study Locations	Basic Assumptions Tests at Significance Level (5%)							
	NERC Test (Randomness)		Wald-Wolfowitz Test (Independence)		Mann-Whitney Test (Homogeneity)		Spearman Rank Correlation Test (Stationary)	
	Statistics	p-value	Statistic	p-value	Statistic	p-value	Statistic	p-value
Astore	0.5561	0.2891	1.1915	0.1167	-1.8898	0.094	1.134	0.1284
Bunji	0.5561	0.2891	0.0438	0.4825	-0.6803	0.2481	0.354	0.03617
Chillas	2.7806	0.27	-0.5786	0.2814	-1.2851	0.0994	1.678	0.067
Gilgit	-0.5561	0.2891	-0.4595	0.323	-0.1512	0.4399	0.3474	0.3641
Gupis	-1.1123	0.133	-0.0397	0.4841	-1.5875	0.062	-1.8943	0.091
Hunza	0.9908	0.1609	1.5306	0.0629	-2.0817	0.087	-2.2729	0.115
Skardu	0	0.5	-0.1079	0.457	-1.2095	0.1132	0.9767	0.1644

DISCORDANCY MEASURES

In regional frequency analysis, the discordancy measure (Di) is used to identify sites with atypical or outlier behavior that may distort regional homogeneity. Following the methodology of Hosking and Wallis, critical values for Di are determined based on the number of sites in the region. In this study, which includes seven rainfall stations, the critical threshold value for Di is 1.91. According to Table 4, none of the stations have a Di value exceeding this threshold. This indicates that all sites are consistent with the regional data pattern and no station is statistically discordant. Therefore, the region does not exhibit significant outliers, and all seven sites are suitable for inclusion in the regional frequency analysis.

Table 3: Critical Values for Di Discordancy Test

No of Sites	5	6	7	8	9	10	11	12	13	14	>15
Critical Values	1.333	1.684	1.917	2.140	2.329	2.491	2.632	2.757	2.869	2.971	3.00

Table 4: Summary Statistics for the annual maximum rainfall totals (mm) and discordancy measure (Di) for seven Northern area of Pakistan

Site Name	**N**	**Mean**	**L- CV** (τ)	**L-Skewness** (τ_3)	**L-Kurtosis**	**t5**	**Di**
Astore	18	43.77	0.2054	0.0448	0.0338	0.0338	1.18
Bunji	18	27.32	0.2742	0.1042	0.1040	0.1070	0.85
Chillas	18	28.75	0.2279	0.2798	0.2830	0.0926	0.78
Gilgit	18	23.62	0.2719	0.3507	0.2382	0.1174	0.79
Gupis	18	26.92	0.2958	0.3311	0.2038	0.2139	0.74
Hunza	9	20.95	0.3069	0.4686	0.5328	0.5177	1.75
Skardu	17	29.76	0.3037	0.1660	0.0784	0.0307	0.91

The summary statistics and discordancy measures (Di) for annual maximum rainfall across seven sites in northern Pakistan are presented in Table 4. The mean annual maximum rainfall ranges from 20.95 mm (Hunza) to 43.77 mm (Astore), with L-moments indicating varying degrees of dispersion (L-CV = 0.2054–0.3069), skewness, and kurtosis across sites. Higher L-skewness and L-kurtosis values for Hunza suggest a heavier-tailed and more asymmetric distribution. The discordancy measure (Di) is used to identify outlying sites within the region, with a critical threshold of 1.91 for seven sites. All stations have Di values below this threshold, with the highest being Hunza (Di = 1.75), which, although close to the cutoff, does not exceed it. This indicates that none of the sites are statistically discordant, supporting the homogeneity of the region and the suitability of pooling data for regional frequency analysis.

IDENTIFICATION OF HOMOGENEOUS REGIONS

The identification of homogeneous regions is a critical step in Regional Frequency Analysis (RFA), requiring careful consideration and subjective judgment. A homogeneous region is defined as a group of sites that, despite differences in scale, exhibit similar underlying frequency distributions of extreme rainfall, characterized by consistent population L-moment ratios. In this study, the seven rainfall stations Astore, Bunji, Chillas, Gilgit, Gupis, Hunza, and Skardu are treated as a single region. To assess the homogeneity of this region, the heterogeneity measure (H) is computed using the method outlined by Hosking and Wallis (1997), based on L-moment ratios and Monte Carlo simulations.

The results, presented in Table 4.5, show heterogeneity statistics of $H1 = -0.62$, $H2 = 0.43$, and $H3 = 0.67$. According to established criteria, a region is considered acceptably homogeneous if $H<1$, possibly heterogeneous if $1 \leq H<2$, and definitely heterogeneous if $H \geq 2$. Since all three H values are well below the threshold of 1, the region is deemed acceptably homogeneous. This indicates that the inter-site variation in L-moment ratios is consistent with what would be expected from a homogeneous region, supporting the pooling of data across the seven sites for regional frequency modeling. Thus, the assumption of regional homogeneity is satisfied, allowing for reliable regional quantile estimation.

Table5: Heterogeneity measures for the region under study

No. of sites	H1	H2	H3
Heterogeneity statistic for 7 stations	-0.62	0.43	0.67

The results, presented in Table 4.5, show heterogeneity statistics of $H1 = -0.62$, $H2 = 0.43$, and $H3 = 0.67$. According to established criteria, a region is considered acceptably homogeneous if $H<1$, possibly heterogeneous if $1 \leq H<2$, and definitely heterogeneous if $H \geq 2$. Since all three H values are well below the threshold of 1, the region is deemed acceptably homogeneous. This indicates that the inter-site variation in L-moment ratios is consistent with what would be expected from a homogeneous region, supporting the pooling of data across the seven sites for regional frequency modeling. Thus, the assumption of regional homogeneity is satisfied, allowing for reliable regional quantile estimation

SELECTION OF BEST REGIONAL DISTRIBUTION:

The objective extends beyond merely identifying the most suitable distribution; it also aims to provide accurate and reliable quantile estimates for each region. To achieve this, the goodness-of-fit statistic Z_{DIST} is computed for all stations to evaluate the performance of five candidate distributions: Generalized Logistic (GLO), Pearson Type III (PE3), Generalized Extreme Value (GEV), Generalized Normal (GNO), and Generalized Pareto (GPA). This approach ensures a robust selection of the best-fitting distribution for regional frequency analysis.

L-MOMENT RATIO DIAGRAM

Hosking and Wallis (1997) recommend using the L-moment ratio (LMR) diagram a plot of L-kurtosis versus L-skewness as a graphical tool for selecting the most suitable regional probability distribution. The LMR diagram exploits the unique relationship between L-moment ratios for different theoretical distributions. Figure 4 displays the L-moment ratios of seven stations in northern Pakistan alongside five candidate distributions. The proximity of the data points to the theoretical GLO (Generalized Logistic) distribution curve indicates that GLO is the most appropriate distribution for the region.

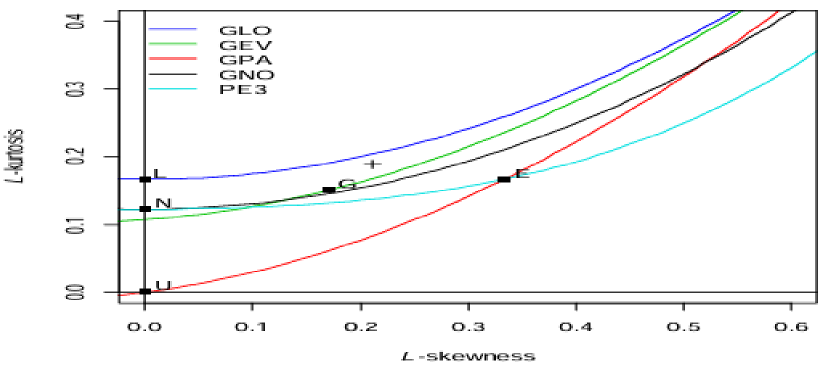

Figure.4 L-moment ratio diagram for the data of seven sites

GOODNESS OF FIT CRITERIA (Z^{DIST} STATISTIC)

Hosking and Wallis (1997) recommend using the L-moment ratio (LMR) diagram a plot of L-kurtosis versus L-skewness as a graphical tool for selecting the most suitable regional probability distribution. The LMR diagram exploits the unique relationship between L-moment ratios for different theoretical distributions. Figure 4 displays the L-moment ratios of seven stations in northern Pakistan alongside five candidate distributions. The proximity of the data points to the theoretical GLO (Generalized Logistic) distribution curve indicates that GLO is the most appropriate distribution for the region.

Table 6 Z^{DIST} Statistics for various distribution understudy (Z^{DIST})

istributions	Z^{DIST}	$Z^{DIST}\|$
GLO	0.34	0.34
GEV	-0.42	0.42
GNO	-0.71	0.71
PE3	-1.25	1.25
GPA	-2.27	2.27

PARAMETERS OF GENERALIZED LOGISTIC DISTRIBUTION

Table 6 presents the parameter estimates of the Generalized Logistic (GLO) distribution obtained using two distinct estimation methods: the L-moment method and the Bayesian approach. The GLO distribution is characterized by three parameters shape, scale, and location each of which is estimated under both frameworks, allowing for a comparative assessment of their performance and stability.

Table 7 Estimated parameter of GLO distribution

Parameters	Α	Ξ	Κ
L-moment	0.9006	0.2429	-0.2333
Bayesian	1.094	0.23334	-0.3299

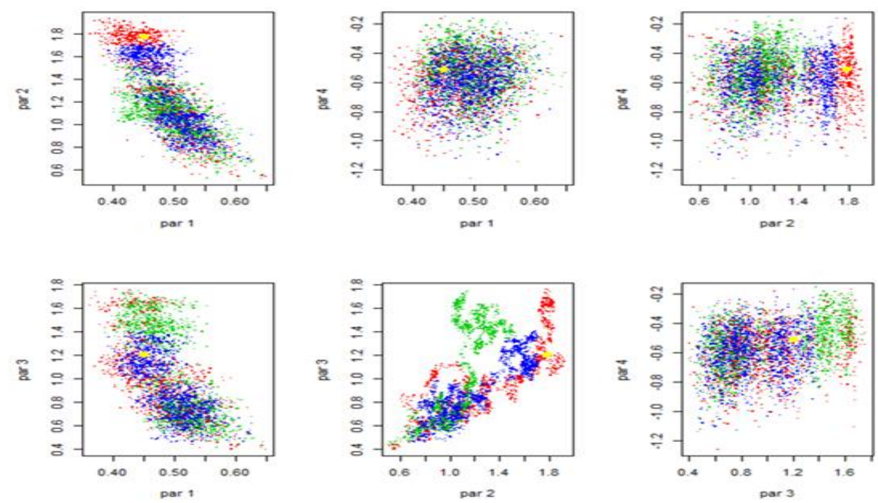

Figure 5 Graphical representation of hyper parameter

ESTIMATION OF REGIONAL QUANTILES

Two different estimation methods L-moments and Bayesian techniques are employed to estimate regional quantiles for the Generalized Logistic (GLO) distribution, with results presented in Table 4.7. These quantiles, corresponding to specific non-exceedance probabilities, provide insights into the distribution of extreme rainfall across the region. However, regional quantiles alone are insufficient for precise local predictions. To address this limitation and improve predictive accuracy at individual locations, at-site quantiles are derived using the regional growth curve method. These at-site quantiles, shown in Table .8, offer more targeted and reliable estimates for future extreme rainfall events at specific stations, thereby supporting better-informed decision-making in hydrological and environmental planning.

Table 8 Regional simulation: sevensites, 10000s imulations Relative RMSE and error bounds for ratio of estimated regional growth curve to true at-site growth curve

RP	**2**	**5**	**10**	**20**	**50**	**100**
F	0.50	0.80	0.90	0.96	0.980	0.990
RQ_{Lmom}	0.701	0.792	0.987	1.121	1.501	2.091
$RQ_{Bayesia}$	0.711	0.811	0.991	1.321	1.621	2.123
RRMSE(L-mom)	0.071	0.039	0.064	0.096	0.171	0.340
RRMSE(Bayesian)	0.060	0.021	0.063	0.097	0.172	0.341
UB	0.204	1.031	1.451	1.801	2.032	2.451
LU	1.021	1.404	1.902	3.452	3.956	5.156

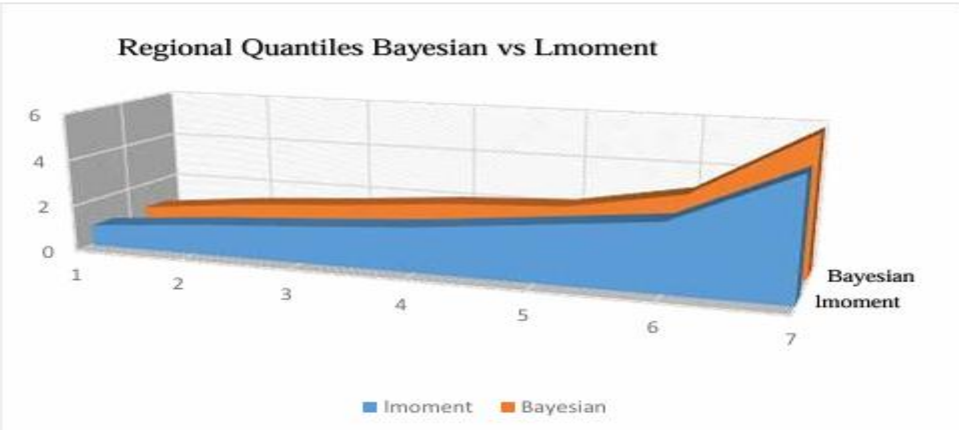

Figure 6 Regional growth curve for the seven stations of Northern areas of Pakistan

ESTIMATION OF AT-SITE QUANTILES

The regional quantiles were derived using the Generalized Logistic (GLO) distribution, identified as the best-fitting regional distribution. These regional growth factors were then combined with the observed mean annual maximum rainfall at each site to estimate site-specific (at-site) quantiles. For high return periods, such as the 100-year event, the estimated quantiles were 2.091 using the L-moment method and 2.123 using the Bayesian approach, indicating close agreement between the two methods. For shorter return periods (T < 10 years), regional quantiles were typically 1–3% lower than at-site estimates, though the difference was not considered practically significant. While the at-site method exhibited lower bias, particularly for lower return periods, the regional approach remains valuable for data-scarce locations. The resulting at-site quantiles provide localized predictions—for instance, in Astore, the expected annual maximum rainfall over the next five years is likely to reach approximately 34.67 mm. As shown in Table 4.8, the Bayesian technique slightly outperforms the L-moment method in estimating at-site quantiles, yielding more accurate and stable results, especially under uncertainty.

Table 9 Estimation of At-Site Quantiles of Seven Stations and compare the quantiles of 18 years AMRS with the average maximum rainfall in the year of 2022 & 2023

Stations	2022 (maximum rainfall in mm)	2023 (maximum rainfall in mm)	Method of Estimation	Returnperiod(inyears)			
				2	5	10	20
ASTORE	32.6	36.1	L-moment	30.68	34.67	42.01	54.82
			Bayesian	31.12	35.49	43.39	57.82
BUNJI	25.7	19	L-moment	19.15	21.64	26.97	33.06
			Bayesian	19.43	22.16	27.08	36.09
CHILLAS	23	30	L-moment	20.15	22.77	28.38	34.79
			Bayesian	20.45	23.32	28.50	37.98
GILGIT	16.7	18.6	L-moment	16.56	18.71	23.32	28.58
			Bayesian	16.80	19.16	23.41	31.20
GUPIS	14.5	13.5	L-moment	18.87	21.32	26.57	32.57
			Bayesian	19.15	21.83	26.68	35.56
HUNZA	19	9	L-moment	14.69	16.59	20.68	25.35
			Bayesian	14.90	16.99	20.77	27.67
SKARDU	31.1	14.2	L-moment	20.86	23.57	29.38	36.01
			Bayesian	21.17	24.14	29.50	39.31

The table compares observed rainfall in 2022–2023 with at-site quantile estimates from L-moment and Bayesian methods for seven stations in northern Pakistan. The quantiles, based on the Generalized Logistic (GLO) distribution, show that the models perform well e.g., Astore's observed rainfall aligns closely with the 5-year return level. Some stations (Bunji, Hunza) had lower rainfall than the 2-year quantile, indicating dry conditions, while others (Chillas, Skardu) exceeded their 10-year and 2-year levels, signaling extreme events. The Bayesian method consistently produces higher, more conservative estimates than L-moments, especially for more extended return periods, due to its incorporation of parameter uncertainty. This makes it more suitable for flood risk management and infrastructure planning. Overall, both methods are effective, but the Bayesian approach offers greater robustness and reliability in extreme value prediction.

CONCLUSION & RECOMMENDATION

The study evaluates the performance of L-moment and Bayesian MCMC methods in modeling extreme rainfall events using Annual Maximum Rainfall Series (AMRS) from seven stations Astore, Bunji, Chillas, Gilgit, Gupis, Hunza, and Skardu in northern Pakistan. The Generalized Logistic (GLO) distribution was identified as the best-fit model based on L-moment ratio diagrams and goodness-of-fit criteria. Preliminary tests—Mann-Whitney U (homogeneity), Wald-Wolfowitz (independence), Spearman's rank correlation (stationarity), and NERC (randomness) confirmed that the data meet the fundamental assumptions for frequency analysis.

Parameter estimation was conducted using both L-moments and Bayesian Markov Chain Monte Carlo (MCMC) methods. Results show that the Bayesian approach, particularly with non-informative priors, outperforms L-moments, especially in small-sample settings where L-moments exhibit higher bias and lower efficiency. Robustness measures, including RMSE and bias analysis, confirm the superiority of the Bayesian method in estimating GLO parameters.

Return level estimation revealed that Bayesian MCMC produces higher and more reliable quantile estimates for return periods (10, 25, 50, and 100 years) compared to L-moments. Predictions for 2022 and 2023, based on 18 years of data (2006–2023) closely matched the observed rainfall in 2018, validating the model's predictive accuracy. The posterior predictive distribution further enhances uncertainty quantification, making it highly valuable for designing hydraulic structures such as dams, bridges, and flood control systems.

In conclusion, the Bayesian MCMC framework offers a more robust, flexible, and uncertainty-aware approach to extreme value analysis in hydrology. It is recommended for use in flood risk assessment and water resources planning in data-scarce regions, such as northern Pakistan. Future studies should explore the integration of informative priors and advanced MCMC diagnostics to improve estimation accuracy and computational efficiency further. Future research should focus on developing error-in-variables models tailored explicitly for post-stratified sampling frameworks, incorporating correction methods such as instrumental variables, simulation-extrapolation (SIMEX), or corrected score approaches to mitigate the impact of measurement error. Additionally, integrating robust auxiliary measures such as median, tri-mean, and Hodges-Lehmann estimators into variance estimation can enhance resilience against outliers and non-differential misclassification. There is also a need for simulation studies under diverse error structures (e.g., multiplicative, systematic, and non-constant variance) and for empirical validation across different domains, including health surveys, environmental monitoring, and financial auditing. Moreover, future work should explore the use of machine learning techniques for error detection and data cleaning in auxiliary variables, as well as the development of user-friendly software tools that implement robust and error-corrected estimation procedures for practitioners. By addressing these gaps, researchers can improve the accuracy, reliability, and applicability of variance estimation in real-world survey settings**.**

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