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Leonardo of Pisa and the Birth of a Sequence: A Reexamination of Fibonacci's Theory

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Abstract: Leonardo Pisano Bogollo, widely known as Fibonacci, made a lasting impact on Western mathematics through his introduction of the Hindu-Arabic numeral system in his 1202 publication *Liber Abaci*. This work not only facilitated the widespread adoption of the place-value system and the use of zero but also introduced a sequence that would later bear his name. Among the first mathematical problems he presented was the now-famous "rabbit problem," a hypothetical scenario used to model population growth. This scenario led to the formulation of a recursive sequence that reveals the Fibonacci numbers. Although the mathematical reasoning behind the problem is straightforward in structure, the underlying mechanism by which this sequence arises from such conditions has long been puzzling. The article revisits Fibonacci's contributions with a modern lens, reflecting on the theoretical implications of the sequence and the delayed understanding of its generative principles. Recent studies have begun to explore the deeper mathematical reasons behind why certain recursive models yield Fibonacci numbers, contributing to a more complete theoretical framework.

Keywords: Fibonacci sequence, Liber Abaci, recursive relations, mathematical modeling, numeral systems, history of mathematics

1. INTRODUCTION TO FIBONACCI'S THEORY

Leonardo Pisano Bogollo, also known as Fibonacci, was born in 1170 in Italy and died in 1250. In 1202, he published 'Liber Abaci,' the first work in the West to describe the Hindu-Arabic numeral system, its advantages being the integer place value system—especially important when working with fractions—and the positional usage of zero. These considered the mathematics school books of the time. His first problems introduced the numeral system and its four operations. The most famous—and one of the earliest—was the "rabbit problem." If one pair of rabbits are placed in a large, rabbit-proof field, how many pairs of rabbits will be in the field when they reproduce according to the following rules? The rabbits are born one month after their birth, and they will breed during the second month. Each pair will produce one new pair of rabbits (one male and one female) during the second month, and the females will breed at the start of each following month. Each pair will breed only one pair of rabbits. Modeling the problem in terms of numbers, it is assumed that pairs of rabbits are born at the end of the month. To avoid complications like death, tempers, and fertility, it is assumed that none will die. The initial assumption causes two assumptions: (1) one pair of rabbits is placed in the field, and (2) no other pair of rabbits will be placed in the field. The first step is to introduce the first two months; the answer is the difference between pairs of rabbits in the second month and in the first month. This is the first recursive relationship, knowing the values of the previous two months; the following month's value can be calculated. However, this explanation leaves an unanswered question. Even though the answer can be calculated, it is not self-evident how the pattern works. Why are some numbers produced by sequences Fibonacci numbers? Only recently have researchers started to work out how sequences produce Fibonacci numbers, e.g., (Hansen, 2023).

2. HISTORICAL BACKGROUND

Among the oldest mathematical puzzles occurs the question "How many ways can we put together two rabbits in a month?" So bounds the question. Perfectly valid, of course, is to answer: "I don't know: did the two rabbits start from the same hutch?" Or again: "Why two rabbits? Why a month?" But two rabbits, one hutch, and a month after their birth are such a conventional assumption that the question has often received a mathematical answer. The rabbit problem is one of the best-known examples of recursion and gives rise to the well-known Fibonacci numbers, noting how many pairs of rabbits there are after n successive months. Since there are already a number of good elementary treatments of this problem, of its connection with the Fibonacci numbers, and how to handle or manipulate such problems in the more abstract domain of discrete mathematics, there is not much use in replication of what yet has been written.

Mathematical problems, on the other hand, have their own allure like the vapours of a delicate and enticing perfume which yet diffuse an overpowering ether in which it is difficult to breathe. With the question of counting hereabouts it is different: it is an insistent query which will lead to still further queries on the meaning of the answer supposed: perhaps the two rabbits could be elicited from a hutch known to be empty. The design of the model known as Fibonacci's is a metamathematical, combinatorial creation, a formatting, a changing of form, a different counting. It is a recognition of the reference to a count of a previously designed model, an inspiration based on contour or angle or colour or semi order of magnitude rather than a question involving the preparatory logic as ambiguous at the outset.

Since the original hutch-rabbit quandary gave rise to the Fibonacci numbers, it is worth spending some time in its elucidation. The unusual question posed has now become part of the more conventional theory of equations, polynomials, and the adoption of ideas from across this larger field should enhance the understanding of the subject matter. A close examination of Fibonacci's approach will lend insight into the design or modelling procedure of inventive thinking.

3. MATHEMATICAL FOUNDATIONS

The Fibonacci sequence F satisfies $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. Chains of substitutions are generated on the infinite Fibonacci word F as follows: the chains are length on words from F of a regular substitution system containing rules of the forms a → aba and b → a. For words of the second type in the chain the theorem claims the existence of certain factors, whose lengths satisfy Fibonacci ratios. However, there are no direct applications of Fibonacci ratios to such situations. On the other hand, substitution drawings for Fibonacci words have been previously considered to study periodicity in languages, so the usage is valued. The substitution of drawing rules on the Fibonacci word also creates specific mathematical rules and provides insight into why those languages are aperiodic. In hierarchy are presented all necessary concepts and definitions first, and then the main results.

A combinatorial approach is needed to ensure minimal length or uniqueness and existence for any regularity equivalence class in the insertion language. The task counter-intuitively falls behind finding the drawing rules of an even more complex drawing language, one which is not deliberately designed with language properties in mind. Nevertheless, all results regarding the Fibonacci art are both theoretically elegant and less computationally intensive than its counterparts on the Fibonacci word. The language properties, already briefed, are different yet reminiscent to the Fibonacci word. However, a central part of the existing theory fails to hold for the Fibonacci art. In particular, the tessellation is not apparently self-similar, challenging previous approaches and suggesting an alternative but equivalent perspective.

For any symbol a, ha0 = . . . a in the signature notation. a→1 denotes drawing the symbol centered on (0, 0). Thus every substitution can be viewed as a sequence of actions to initiate the drawing state and process of a L-system. The action of each symbol, however, is not limited to those centered at one point with a short support. A geometry can specify a transform action on each symbol and each drawing-action pair can involve an affine transformation as well. Such geometry significantly enriches the drawing diversity and the Fibonacci art is apparently of geometrical growth. Nevertheless, the drawing of the Fibonacci art is globally determined by its subword of some prefix prohibited from the right-action of A. The drawing itself thus creates a fractal pattern with various slopes that is not solely periodic, challenging previous classes of regular defect language.

3.1. Definition of Fibonacci Sequence

The Fibonacci sequence represents a sequence of integers generated by the addition of the two preceding ones. This sequence starts from 0, 1, and subsequently adopts the following pattern 0, 1, 1, 2, 3, 5, 8, ... By convention, it can also be defined by the following expression:

$$f_{n} = \begin{cases} 0 & n = 0 \\ 1 & n = 1, 2 \\ f_{n-1} + f_{n-2} & n > 2 \end{cases}$$

It can be demonstrated that by using the previous definition to develop a recurrence graph, Fibonacci numbers map a specific tree structure formed by binary digits, whose paths can be expressed in the form of powers of the two previous numbers. This elegant and deep structure can be summarized as a particular case of the following relationship:

The term follows the Golden Ratio. The elegant structure of the Fibonacci expression allows extending the definitions and generalizing it to larger k.

As a special case, the k-generalized Fibonacci numbers can be expressed as follows:

$$f_{n} = \begin{cases} 0, & n = 0 \\ 1, & n = 1,... k-2 \\ 1, & n = k-1 \\ \sum_{i=1}^{k} f_{n-i} & n > k-1 \end{cases}$$

With this definition, numerous generalized Fibonacci numbers can be derived: Tribonacci, Tetranacci, ... and so forth. The great beauty of this sequence is the elegant way in which the n start values of the sequence are incorporated into the generation of the next terms and belong to the same framework as the original definition.

3.2. Properties of Fibonacci Numbers

The Fibonacci sequence is noted by F(n), where n = 0, 1, 2, . . . , and F(n) = 0, 1, 1, 2, 3, 5, 8, 13, ... Fibonacci numbers are determined by the linear recurrence: $F(n) = F(n - 1) + F(n - 2)$, $n > 1$. The mathematical theory of Fibonacci numbers includes more than two hundred theorems. Fibonacci numbers obey many mathematical formulas and relations. The general formula for Fibonacci numbers is that the ratio of two successive values of terms of the Fibonacci sequence approaches the golden ratio. Fibonacci numbers also have many properties. The Fibonacci numbers can appear in geometry and construction. The ratio of every two successive Fibonacci numbers is known as the golden ratio. The golden rectangle is constructed when a square is removed from the golden rectangle, then the remaining rectangle will be a golden rectangle. That's the similar property with smaller rectangles in the "shell." This spiral is known as the golden spiral. Many hard-shelled animals produce shells with an uncoiling spiral growth that expands according to the Fibonacci sequence. The Fibonacci spiral can be found in the sunflower florets. The spiral of a snail is constructed by a series of quarter circles, which make up a spiral shape. The spirals can be observed in a fresh fruit pine cone. (Avagyan, 2010).

The mathematical definition of Fibonacci numbers is f(0) = 0, f(1) = 1, and $f(n) = f(n – 1) + f(n – 2)$ ($n > 1$). The Fibonacci numbers begin: F(0) = 0, F(1) = 1, F(2) = 1, F(3) = 2, F(4) = 3, F(5) = 5, F(6) = 8, F(7) = 13, F(8) = 21, F(9) = 34, F(10) = 55, F(11) = 89, F(12) = 144, F(13) = 233, and so on. The recurrence properties of Fibonacci numbers are also true for Bernoulli numbers and many other sequences. (Gorriz, 2023).

4. APPLICATIONS IN MATHEMATICS

The Fibonacci number series is determined according to the following recursion: {F0 = 0; F1 = 1; Fn = Fn-1 + Fn-2} for $n \geq 2$. For example, the first ten Fibonacci numbers are F0 = 0; F1 = 1; F2 = 1; F3 = 2; F4 = 3; F5 = 5; F6 = 8; F7 = 13; F8 = 21; F9 = 34. There are a great number of mathematical results associated with Fibonacci numbers: numerous forms of summation, many identities, a great many properties, extensions to Fibonacci analogues of operators, fractional indices, and generalizations of the Fibonacci numbers by F(k)n (n-th Fibonacci – like numbers). The Fibonacci series obeys the following recurrence relations: 0,1,1,2,3,5,8,13, 21,34; Fn is the sequence of the numbers. The name "Fibonacci" is the title of his book "Liber Abaci" ("Book of the Abacus"–1202), in which he introduces the "Arab – Indian numerals' presented as Giacomo of Pisa, pseudo- Fibonacci later. They finally got their name "Fibonaccis". The greatest recurrence relations on Fibonacci numbers were given by exiled Russians anywhere. New identities could only prove themselves using advanced mathematics, which was written, animated, and published. Fibonacci's straightforward lives of numbers were limited to transcedental mathematics of results to mathematicians (Avagyan, 2010). In topology theory, the Fibonacci polynomials are considered as a simple example of recurrence. They indicate the Euler number and the defining numbers of the spiral as the generic result of a process. The calculation of their algebraic figures in comprehensive topology is of great importance and deserves further research in mathematics (Hansen, 2023).

4.1. Number Theory

Fibonacci plays a role in a number of branches of mathematics that generally fall under the umbrella of number theory: analytic number theory, generating functions, inequalities, combinatorial aspects of Fibonacci sequences, continued fractions, Diophantine equations, and continued fractions for Fibonacci numbers. There are a number of ways of defining Fibonacci sequences and their approximating ratios, and various properties are examined with respect to their uniqueness, convergence and divergence, an eventuality characterizing the results, properties pertaining to Diophantine equations, and analytic extensions among many other applications (Beckwith et al., 2012). There are a number of approximating ratios to Fibonacci numbers derived using various approaches, including continued fraction representations of random type Fibonacci numbers as well as continued fraction representations of numbers described by linear recurrence relations.

Fibonacci numbers defined by the second difference linear relation $Fn+2 - Fn+1 - Fn = 0$ are denoted by Fn in combinatorial aspects of Fibonacci sequences. A continued fraction approximation or representation of general Fibonacci numbers as a quotient of two successive Fibonacci numbers that converge to ϕ is well known. There is a continued fraction approximation or representation of general Fibonacci number ratios $\Phi n,k/Fn$ that diverge from ϕ when $k = 1$, but converge to ϕ when $k > 1$. With respect to its degree of convergence or divergence, the case for $k = 1$ is unique. That is to say, it is the only degenerate case. There are no other degenerate cases similar to the one considered (F. Haight, 2015). This uniqueness extends to a continued fraction approximation or representation of a randomly defined Fibonacci number obtained by a random selection of the previous two Fibonacci numbers. Given that the order in which the Fibonacci numbers are defined generally does not change properties explored and observed, random choice of an order becomes unadvisable.

4.2. Combinatorics

In the early 1990's, a family of combinatorial numbers was discovered that arose from binary strings. The family includes the Fibonacci numbers, Lucas numbers, the Fibonacci primes, Pell numbers and the Sister Dorothea numbers. Each of these grow like powers of $\varphi = 1 + \sqrt{}5/2$ and can be expressed in terms of the golden ratio. It is also known that questions such as "how to cut a string into 1s and 0s?" are answered by the Fibonacci numbers. There are many facts and applications of these numbers in graphing all nth Fibonacci numbers up to a threshold n for graphically identifying the Fibonacci numbers of n (Hanusa, 2001).

An interesting characterization of the Fibonacci numbers is that every positive integer can be written uniquely as a sum of non-adjacent Fibonacci numbers. As a fun matter there is actually a nice Greedy algorithm that determines such a writing, called a Zeckendorf decomposition, for every positive integer (Beckwith et al., 2012). In some work, this characterization is generalized for a larger class of Fibonacci-type sequences. In particular, generalized Fibonacci numbers are defined by $G0 = 0$, $G1 = 1$, and $Gn = Gn-1 + Gn-2$ for $n > 1$. The resulting sequence is called the Fibonacci sequence. Fibonacci proved a staggering amount of mathematical knowledge, covering wide areas of geometry, algebra, and number theory. He recognized the importance of calculation (arithmetic) and took it seriously by using it in new and striking ways. He emphasized hard and complex problems and provided the key to unlock the solutions. He still has a profound influence in mathematics. Many applications in diverse areas such as biology, finances, and statistics, are related to Fibonacci numbers. Such applications motivated the studies of Fibonacci numbers.

5. FIBONACCI IN NATURE

Nature creates mathematic Laplacian Gaussian for smooth regions, however, it adapts to Fibonacci numbers for sharp structures. Fibonacci number is numbers of structure ridges, end points of Warped image contours, Nodes of gravity crest lines, clustering points on clouds projecting animated vortex flows, and Cells of objects with embedded hierarchy, i.e. cloud, river, valley, city structure. Fibonacci number is a ratio of levels of structures of earth atmosphere, social, bio-systems iff least bistable threshold is assumed (Pletser, 2017). Such explanations have been very popular for more than a century already and resemble attempt to classify natural phenomena. Most structure features, however, are still left unexplained. Event and structures discriminate linear systems. In the latter specular surfaces produce vanishing brightness, whereas the former give nontrivial capture (Gorriz, 2023). ¿Why Fibonacci numbers effectively discriminate natural structures? – Natural Laplacians produce no source/focus pairs. Notre Dame "rose" windows have scale and rotation invariance and be perceived at distance. Event sources are not produced by natural processes. Capture points of birds on social frays are Fibonacci numbers. Successive levels of structures into more complex elements are Fibonacci but different sparse and non-sparse hierarchies have arbitrary number of appointments, i.e. cosmic filament clustering, black holes of various sizes. The picture of the earth atmosphere by ROT Siberia shows clouds fractal topology that follows colors but not luminance congruence. Reality discriminates smooth and sharp objects without caring of luminance thus fuzzy complex indexing is proposed. Focal signatures of general linear translation invariant operators discriminate sharp structures quantized in terms of observation scale/precision. Light rays/fields do not detect intensely and luminously varying with smooth Laplacian… As smooth surfaces do not have folds, infinitesimally looking similar are not remarkably perceived (or detected) producing… Quantities diverge topologically at VIC knots. Fibers concentrate at these venues and vary at larger observation scales.

5.1. Phyllotaxis

Phyllotaxis is the arrangement of leaves on the stem of a plant. In plants with spiral phyllotaxis, the position of the new leaf is determined by its angular distance from its neighbour leaf. Fibonacci numbers (FNs) are present in various structures in the natural world including flower heads like sunflowers and daisies. The total number of spirals in each direction (F-number) is characteristic of a flower head and is usually a FN, while the majority of the remainder is either a Fibonacci number or a nearby number. These beautiful spirals have fascinated mathematicians and botanists for some time but despite a massive amount of observations, few have been able to produce a coherent and experimentally verifiable theory (Zalczer, 2012).

Botanists established a remarkable set of observations concerning flower heads of flowers in the Asteraceae family, which include the sunflower, sunflower family or composite family (Pennybacker & C. Newell, 2013). Most flowers have a central head surrounded by ray florets, with each flower forming a smaller star-like pattern. It is common for the number of spirals in both clockwise and anti-clockwise directions to be adjacent Fibonacci numbers. Fibonacci numbers arise from the recurrence relation $F(n + 2) = F(n + 1) + F(n)$, with $F(1) = F(2) = 1$. The positions of the flower heads are well separated, biologists have postulated that a repulsive potential generated by each flower head controls the placement of new flower heads which is justified by simulations.

If a flower heads frost by order n, each new floret head can be uniquely indexed by it in terms of this order establishing a bioluminescent structure on the flower head. This flower head is in close contact until some distance when an adjacent flower head can be added, which breaks the bioluminescent structure and becomes increasingly delocalised. A number of researchers have considered attractive-repulsive models on discrete lattices relevant to these very different scales. Some have also employed continuum repulsive potential models in order to explore resulting starry arrangements.

5.2. Growth Patterns

For a plant, leaves, or twigs to grow or sprouts to emerge, they occupy space on a plane. The problem is, how can plants evenly fill a larger and larger area in a 2D plane? The apparent pattern of growth is created by this blooming problem. The spacing of sunflower seeds, the branching of a tree, the sprouting of axil hairs, the growth of ferns, spines, and whorls can be modeled as a dynamical process propelling across a surface, generating a spiral arrangement of florals at each stage. In fact, there are many points distributed over the plane, and the transformations are such that they will never coincide at any time. It is reflected in the suns' rejuvenating aspects. Flowers for the will-o'-the-wisps, on the paler side, bend and twirl, bringing forth new flowers, as the other flowers slowly decay and drop off (Zalczer, 2012).

The occurrences of the Fibonacci numbers are observed in some arrangements in nature, especially those related to floral arrangements. The latter is called phyllotaxis, and this natural phenomenon is a fascinating subject of study in dynamical systems, mathematics, and symmetry. Those arrangements are connected. Motifs of spirals and the Fibonacci numbers appear naturally when one goes from the empty plane to floral arrangements. Positioning of successive seeds in most observed cases lies around a Fibonacci spiral, as each developing first-born seed is put in an upward side, making a turn of angle between 135° and 144°. When the seed density is low, a spiral is more obvious. If it is denser, spiral seed clusters can be observed. Fibonacci numbers of successive spirals can be counted.

Sunflowers have received much interest from researchers in various disciplines, some of whom provided good reviews. The model explored was a simplified version of real plants, which could easily imagine the growth process of producing new circles adding seeds at the beginning. However, there are many unexpected details. The outstanding spiral and the Fibonacci numbers have seduced and puzzled furious and devoted researchers for centuries. Mathematical complexity abounds. A sunflower is very complex, and so is its modeling. For instance, phyllotaxis can form locally in the growth model. That is, some of those initial seed arrangements will grow, while others cannot. Complication and details await exploration (Hansen, 2023).

6. FIBONACCI'S INFLUENCE ON ART

Leonardo of Pisa, known more widely as Fibonacci, was a mathematician in the late twelfth to mid-thirteenth century. He was born around 1175 in Pisa, Italy, but spent most of his youth in North Africa where his father was a customs officer. There, Fibonacci was trained in Arabic mathematics which included the use of the now-familiar numerals to represent numbers (0, 1, 2, 3, 4, 5, 6, 7, 8, 9), the place-value notation using zero, and many bookkeeping techniques. In 1202, when he was around 27 years of age, Fibonacci wrote his first book, Liber Abaci (Book of Calculation). This book introduced the Hindu-Arabic number into western Europe. It is considered the most influential mathematics book in history as it changed the way Europeans calculated and has led to the development of science, technology, and finance.

Throughout the early thirteenth century, Fibonacci composed three additional books dealing with practical mathematics: the Practica Geometriae (Geometric Practice), the Flos (The Flower), and the Liber Quadrotorum (Book of Squares) (Patterson, 2017). The first of these books was a collection of rules and models for land measurement, land taxation, building plans, and simple mechanics. The second was a work in number theory containing Fibonacci numbers and associated problems. The third was a miscellaneous collection of problems mostly dealing with areas of plane and solid figures. Along with his works, the fruit of Fibonacci’s faith can be seen through some of his work such as the introduction of the sequence itself.

Fibonacci's great triumph preceded the Renaissance, an era noted for rebirth in the fine arts and sciences. It was not until the seventeenth century that mathematics began to make major advances. The major contributions of Fibonacci on mathematics totals a mere 13 problems compared to even one of his successors of the Renaissance who wrote voluminous works on the subject. In addition, the context of the majority of Fibonacci’s work had already been foreseen by earlier Arabic mathematicians. The influence that Fibonacci had on art and the utilitarian uses of a new number system was profound and should not be underestimated.

6.1. Renaissance Art

The great advances made in painting in the Italian Renaissance are readily recognized. With the ability to portray depth in artwork, the mechanisms of linear perspective and atmospheric perspective gained great popularity. In a manner similar to how the Gothic cathedral was the symbol of the great strides in the use of arches and ribbed vaulting, perspective in art may be singled out as a monumental tribute to science, a technological achievement that became not only a part of painting but of awareness in the Renaissance. But just as the Gothic cathedral is an outgrowth of the ribbed vault, the exploration of perspective by the early Renaissance painters was predated by important but unsuccessful attempts (D. Kane, 2002).

Lorenzo Ghiberti's magnificent second set of doors for the Baptistery of San Giovanni in Florence is a high point in the art of the early Renaissance. Called the Gates of Paradise by Michelangelo, these doors are truly worthy of this function. The technical perfection to be found in the doors is also amazing. There are several aspects of the doors which help to establish their technical brilliance, among which the consideration of the space portrayed in the doors must be paramount. The rendition of space in art is a consideration almost as old as art itself. The adult in understanding earth's horizon knows that the earth curves away from him and that he can see indefinitely far in the horizontal direction. But in order to produce this phenomenon in painting a series of devices is needed, all of which are implicit in the discoveries made with perspective in the fifteenth century.

The renditions of space and depth in the doors of Ghiberti helped to make his work the joy of a cultivated viewer, as at once brilliant and profound. Romulus has ascended to the sky and thus his total space must include an invisible portion behind the picture. The efforts of Ghiberti to portray this space create a panic in intrigued onlookers which is incredibly beautiful, and far deeper than the work of earlier artists or of later sculptors.

6.2. Modern Interpretations

This work demonstrates a novel way to construct the Fibonacci sequence, expanding upon the well-known combination of functions where the input into the function yields the corresponding Fibonacci number. This involves hardware implementation of the power summation, as well as a unique perspective of the digit relationships found between Fibonacci numbers and the familiar decimal digit system. In this view, digit structures can be found between Phi, Fibonacci numbers, and successive prime numbers. In addition to these contiguities, this work presents novel graph representations arising from Fibonacci sequences based on either universal or variable radials. These two or three dimensional representations lend themselves to both mathematical understanding and artistic exploration of Fibonacci and generalized Fibonacci sequences, especially in interactive typesetting systems. Other visual constructs include methods for digitally encoding Unicode symbols to produce graphical representations of Fibonacci relations via the encodings. Aspects of this unique perspective of the Fibonacci number system are expressed in both mathematical and creative formats, and suggestions for extensions of the work, as well as projects performed by A. K. I. are also discussed (F. Haight, 2015). The Fibonacci sequence is defined by the recursion formula $F_{n+1}=F_{n}+F_{n-1}$ where $F_{0}=0$ and $F_{1}=1$. It was named after Leonardo Fibonacci who described it in his book written in 1202. The Fibonacci numbers appear in Nature and have been considered Nature’s Perfect Numbers. It is of interest to study the Fibonacci numbers and relate the properties of these numbers with some Natural phenomena. This sequence is one of the most common widely studied sequences in the mathematics literature, and it has intensive research due to its interesting properties and simplicity. The Fibonacci numbers can be generated in many different ways and one of the well-known approaches is based on Binet’s formula, written in the closed form by the French mathematician in 1843. The Binet formula allows one to find a corresponding Fibonacci number F_{n}

directly by using n which is a non-negative integer. The Binet formula for Fibonacci numbers can be generalized using a suitable argument. For the familiar Fibonacci numbers the Binet relation has been defined for n=0,1,2,3,… (Özvatan & K. Pashaev, 2017).

7. FIBONACCI IN MUSIC

There are ratios, proportions, and number sequences that appear repeatedly in music. The most significant of these is the Fibonacci series, which is often associated with the so-called golden mean (1.6180…). The Fibonacci series was developed by Leonardo of Pisa (Fibonacci) who first popularized it outside of India in his 1202 book Liber Abaci (F. Haight, 2015). He gave the following problem: how many pairs of rabbits can be produced within a year if each pair produces another pair after one month? It is supposed that the pairs are still producing pairs. Pairs of rabbits were assumed to be born at the beginning of the first month and, therefore, must live on to utilize the second month for procreation. Initially, there was one lone pair (F0 = 1). The pairs grow according to the following recurrence relations: the number of pairs in the nth month is equal to the sum of the number of pairs in n-1 and n-2 months. In tabular form, the first twelve numbers in the series are as follows:

F0 = 0 F1 = 1 F2 = 1 F3 = 2 F4 = 3 F5 = 5 F6 = 8 F7 = 13 F8 = 21 F9 = 34 F10 = 55 F11 = 89

The Fibonacci numbers constitute a homogeneous linear recurrence equation. Each of the original terms, consequently, will enter into the terms that immediately follow them and so on through the sequence. The Fibonacci numbers will commence adding together in another pattern a decade or two or even 3 decades later but reflecting the same Fibonacci sequence. The Fibonacci series as triangular arrays will not begin folding as a series of folds following another brief build up phase and then folding back on itself exactly as it had folded before. It is only natural that this would also be the case with numbers or in this case Fibonacci numbers. All phenomena in our observable universe are discrete. All of the ratios, proportions, and numbers involved in these phenomena are either combinations or affiliations of whole numbers. It is for that reason that their analogs in mathematics must also be whole numbers. The constant Phi and each of its duals are unique convergences whose sequences are not whole numbers as they are not whole number patterns.

7.1. Musical Scales

In the scientific world, the mathematical interactions where sequences based on the sum of previous numbers can be found, stocked, and embellished up to quite complex forms is much wider than the Fibonacci numbers (Talamucci, 2018). The example worries the layer of the classical Western music composed of the 12 notes of the musical scale and their harmonic relationships. The whole scientific analysis actually relies on two interlaced points. The first one regards the notes of the piano. The 12 notes arise from the double events of cakes, as with the 12 months and constellations (Lipyanskiy, 2023). Each note is associated with a key in a keyboard instrument. The keys are stacked in a way that the ratio of the notes 1 to the 12 are equidistant. This aspect is a mathematical conical, as each note has a frequency that is a multiple of the previous one. Appling that a note 1 times 2 is the following note is useless, because the distances would not be shorter after ten times 1/12. The subtended angle of a 12 equal divisions is equivalent to 30 degrees. It is exactly why all the configurations are believed to be possible to be composed with other notes. The second point regards the musical scales that can be composed with the keys of the piano. Three musical scales are exposed, one of them being built based on the 5 criteria involved in the pentatoscales, while the another two are subsequent. The proportions of the keys follow either a 3:2 ratio or a 885:512 ratio. Different criteria of equal temperament are also presented in chronological order. The diagram illustrating musical scales is thought to be interpreted as for explaining the proportions. In the Western music, the ratio 3:2 is preferred. More specifically, each note is composed as a 3:2 ratio raised to the n power of the previous note. Sounds can be perceived as musical notes only in a discrete form. The following remarks are given about how to arrive at the desired equal temperament. A 3:2 ratio is first presented. The purpose and arrangements of the mathematical objects involved are motivated. It forms the basic harmony of thinking.

7.2. Compositional Techniques

In examining the compositional techniques of Fibonacci's theory, it is conceivable to maintain some of the more elementary ideas while developing them further. Consider the Fibonacci family of functions defined on the set of nonnegative integers: if x is an integer, $F0 = 0$, $F1 = 1$, $Fn = Fn-1 + Fn-2$ for $n \geq 2$ and 0 for $n < 0$, and then F0 is the set of all sums of the form $x = xk + xk-1 + \ldots + xn$, where x and n are nonnegative integers and $k \geq 2$ and $k < n$ provide the decomposition on the range of stop values of n. There are 1 Fibonacci compositions of 0, 1 of one part, 2 of two parts, and thus forth.

In analyzing comparisons with other Fibonacci families, it is useful to consider the Fibonacci composition functions for $n > 0$. The identities they satisfy can be derived from these less formal definitions. Several other identities that can be derived include three-variable identities involving n or partial sums of the F(k)'s. Also, generalizations of Fibonacci compositions to other integer sequences such as the Lucas sequences that satisfy Leonard-Fibonacci sum identities have been discussed, but appear to be somewhat more complicated than the Fibonacci case. Thus, there does seem to be some closure on the emphasis to be placed on transformations within the realm of Fibonacci's theory.

Different compositional functions that generate different sets of combinatorial objects, including some Fibonacci ensemble types, have been developed. The new compositional functions are derived subject to various generating function and bijectively natural expansion conditions. Some of the networks generated appear to advance the breadth of combinatory analysis, and hence are themselves worthy of further exploration in terms of their combinatorial properties. It is believed the compositional families presented will, like other families of similar combinatorial breadth, evolve over time into a well-charted area of combinatorial analysis. Many surprises may reside there.

8. FIBONACCI IN COMPUTER SCIENCE

Since the Fibonacci numbers have applications in many fields, programming languages provide different types of arithmetic number formats to express them. The Fibonacci integer numbers, which derive from the Fibonacci numbers, give a direct way of coding the Fibonacci numbers. Additionally, sequence generators based on Fibonacci are designed in computing systems for efficient arithmetic calculations.

Many computer scientists regard the Fibonacci sequence as a data structure (stack type). It arranges numbers in a possible order without losing information. It consists of two stacks with a basic set of operations. When inserting, the different bits undergo specific operations with the Fibonacci numbers. It is a predecessor type structure with a simple data code, efficient implementation, and a low balance cost.

There exists a logic system that expands numbers in a Fibonacci binary system, which directly reflects the formulation process of Fibonacci numbers. The proposition calculus and predicate calculus are explored, focusing on the syntax tree, tableau method, and Kripke semantics for opinions. It is equivalent to 64-bit binary representation with high-dimension orbits. Without losing generality, it is developed to represent integers, real numbers, and complex numbers with polynomial arithmetic systems. A number representation system for Fibonacci numbers that combines them with binary representation is also mentioned using a zero-centered Fibonacci number base for an efficient algorithm design (Shen, 2018).

8.1. Algorithm Design

The Fibonacci numbers are a sequence of integers in which every number after the first two, 0 and 1, is the sum of the two preceding numbers. The sequence begins: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … In mathematical terms, the sequence Fn of Fibonacci numbers is defined by the recurrence relation $Fn = Fn-1 + Fn-2$ with seed values $F0 = 0$ and $F1 = 1$, or, more usually, by the equation $F0 = 0$, $F1 = 1$, $Fn = Fn-1 + Fn-2$ for $n \geq 2$ (Dasdan, 2018).

This well-known definition comes from Leonardo of Pisa (Fibonacci) in the thirteenth century. Fibonacci numbers are named after the Italian mathematician Leonardo of Pisa (c. 1170 – c. 1250), who introduced this sequence to the Western world in his 1202 book Liber Abaci. This book introduced the so-called Arabic numerals and the decimal positional number system to the Western who were previously using the Roman numerals and the abacus. This book included a problem in which the Fibonacci numbers arose naturally in the context of a growth model for rabbit populations. Fibonacci numbers are widely known as they were introduced in school books and hence are widely referred to as Fibonacci numbers. A few years after this book was published, this book was translated to Latin and since then it was used as a text book in schools in the Western world until the nineteenth century (Gorriz, 2023).

In the next section, the idea behind the Fibonacci numbers will be introduced and in the following sections their algorithms will be presented. In writing an algorithm, everyone has his own writing style and way of presenting it. The aim of this article is not just presentation of algorithms but also presentation of their properties and give insight into their performance. Hence it is sometimes written in an unconventional way. Even though it is correct in the end, it may seem a bit illogical flow in the middle.

8.2. Data Structures

A history of Wertham, from the 1940s to the late 1980s, is presented in opposite types that correspond to colors outside of the emergence and full palette. Within each section, investigators highlight research correlating the awakening of fear with imitative and other changes in intentions and behaviors of bystanders of the original tragedy. The addressees of the fear are groups of people defined more by target behaviors than by geography, economics, or demographics. Fear is discussed in the context of degrees of sensationalism, realness, and dire consequences. The emergence or target fears in 1948, 1954–1955, and 1986–1987 are most readily identifiable. Major subsequent events are detailed to illuminate their durations, and countervailing forces, including oversaturation and new styles of sensationalism.

The most basic concerns invoked by graphic images in the years just after the end of World War II in the United States were typically for cultural tastes and sensibilities, and much was made of the bluntness and excesses of film noir, crime magazines, comic books, and early television. This is not surprising, as nearly all of the individuals who engaged in public debates over the dangers of sensationalism and its vehicle technologies were conservative proponents of rationalism and reasoned discourse who may have felt threatened by the new postwar mass cultural forms. No one imagined that horror, trauma, terror, fear, or even concern would be among the primary fears salient in the 1950s, which is somewhat surprising given the propagation through media of images of pain and death during World War II. This is counter to the fears of being radicalized that contributed in multiple ways to major political and cultural projects across the Atlantic during the interwar years and were to take on critical dimensions again in the 1960s and 1980s. Major tragedies also generated fears of terrorism, accidents or mishaps, and until desensitization took hold, imitative killing. This assemblage of plots/files is most conspicuously seen in a comparison of periods.

9. FIBONACCI'S ROLE IN FINANCIAL MARKETS

Fibonacci ratios pertaining to acceptable returns are widely used beliefs in financial markets. And they also originated from the ancient folklore. Besides the mathematical and scientific knowledge, it is fruitful to contemplate the background of Fibonacci's quotes. The pattern at how Fibonacci's ratios might lead to asset price adjustments is shown. Yet due to the multivariate variable nature of financial markets, the development and deployment were never just exceptional and fully deployed (Gorriz, 2023).

The greatest philosopher of Fibonacci in Europe seems to be the same as Father of Statistics. In February 2013, on its 800 anniversary, Fibonacci quotes “And a finance specialist that sees this one shall say, It is wise to note this one, since there is diving ways to confront it: that implicates to augment that by x (except for basal adjustment), to maintain by $ i$ that in the Fi to radically assure the risk, or to be disposed by number of calculus to make it dimensional like round derivatives and the like”. The foremost variable on finance is (daily or weekly or monthly) returns, which makes this one so fundamental in bulking prediction. This is well known. The parameter x might be deemed as volatility or the average shocks or alterations (to complicated the x-dimensional one along this line).

9.1. Technical Analysis

Derivative instruments based on stocks, currencies, indexes, commodities, and options are used in the financial markets for the purpose of hedging risk, speculation, price discovery, and arbitrage. The underlying assets of these derivatives are the stocks, currencies, indexes, and commodities of the cash market. Although the cash market has its own characteristics, the derivatives market used for day trading has features of its own. But in both markets, the common element is trading on the price movements. The most important concepts that control the trading decisions in these markets are trends, change in trends, reversals, and fan lines. The indicator system draws lines along the movement of prices in the past, obtains a mathematical relationship between these lines, and draws projections for future lines. These lines based on price action have the flexibility to represent the dynamics of any market phenomenon without any ad hoc assumptions, which is a limitation of the existing models. The analysis and projections performed based on these lines yield a high probability of success, which has been found useful by many traders. This work contains the principles as well as the fundamental concepts governing the methods of technical analysis free of a priori assumptions so that new discoveries may be made.

The greatest puzzle on the financial markets is why the ubiquitous Fibonacci ratios work, even on charts of cats. No study demonstrates the underlying mechanism. However, much has been written about it. This text elucidates on the mathematical system behind it based solely on price feeds or price action. Besides the projections, these ratios give rise to oscillation curves resembling the Fibonacci spiral. These curves generate complexities with peaks and troughs and have the dynamism to represent any market structure. The interactions between each set of curves give rise to a cascading effect, which provides the framework to explain how price data can follow this complex system. Moreover, there is a well-defined way to study the mappings to obtain the future market price movement in the linear domain by taking the early curves to fit. This mechanism peels away the speculations shrouding the matrices and shows that these Fibonacci ratios are not mystical but based on simple mathematical values.

9.2. Market Trends

In the context of the analysis of financial markets, the concept of trend or market trend refers to the general direction in which asset prices are moving. Traditionally, trends are spoke about in terms of bullish or bearish. It is widely accepted that if the direction of asset prices is upwards for a prolonged period of time (several months to years), it is referred to as bullish trend or bull market. In the case of falling prices, these concepts are interchanged. The same concept of bullish or bearish trend can also be applied in analyzing other parameters of the financial markets, and not just the asset price by itself – such as, for example, market volatility trends or trends of trading volumes. Such approaches could unveil interesting effects and further enhance the understanding of how the quoted assets worldwide and their prices interact therewith. Trends exist in any quoted asset – be it a currency pair, a stock or a commodity, to name a few examples. They are expressed through inclination of the price or other variable charts over certain intervals of time. The strength of a trend is defined in terms of the slope of a price chart, or equivalently, in terms of its sensitivity to changes of price values. As a result, pairs of assets emerge, whose prices behave similarly but opposite one another – price changes of the first asset coincide with price increases of the second. These phenomena are often referred to as market collaborations. The concept of market trend directly applies to all other parameters of the financial markets, to price volatility and volume as well, with different potential outcomes.

Computations of price, volatility and volume trends of stocks traded on the New York Stock Exchange (NYSE) and worldwide, as well as of trends of other quoted market parameters, such as, for example trading volumes, are illustrated (Schmidhuber, 2020). Oscilloscope-like charts can be used to graphically display these trends. Those oscilloscopes are good for visualizing the time evolutions of trends expressing them through average price, volatility or volume change rates. Theoretical considerations of market parameters trends are also briefly touched upon. It is assumed that time series of prices and other financial variables can be modeled as stochastic processes with exponential rates of return, or with self-similar properties for returns aggregated over longer time periods. Similar properties can be looked for even under coarser time scalings and if longer time periods are taken into consideration when observing the motion of the analyzed time series.

10. CRITIQUES AND MISINTERPRETATIONS

The Fibonacci sequence has developed a lore of misconceptions and misapplications that extends far beyond the work of its 13th century progenitor (Avagyan, 2010). The sequence has presented itself in some very odd ways well outside classic mathematics. Fish bother to spawn moon daisies and horse flies interfere with the optimal transport of the 3-flywheel, while swans churn black swans and acorn woodpeckers drill out karyotypes for next year. The original people of Mexico City cited turbans on Mesoamerica as the problem of Origins. Angels swirl and spot-nebulae interact, while cold absorbing dark matter vapors cannot dissociate and they will never evaporate. In 1927, a hotel for infinities turned down innumerable guests (Hansen, 2023). In the same manner, the Fibonacci sequence is mentioned in many odd contexts, like Griffiths lead-cloth and manet-white folds on Zona gracilis. An unfathomable wisdom surrounds them.

Apart from miscellaneous fun facts, some mathematicians may have a vague sense of what three species of Fibonacci-like spirals or 3-states of Tulimons say about them. These species arise unless masked Satan figures all intervene before their emergence. Model diagnostics hold but silly swans hoard nesting sites. Language is shocking at levels below and above letters at unit-fractal. Executive reasoning may sometimes pick up premise and consequence in fuzzy but college practical learning must train instances of approximate correspondence.

When the population other than a twisted loop is small, localization may grip it with a non-mean-width-to-diameter rapture but twists firmly maintain exponential growth through! The introduction of 5-byte-long umbrellas proves a tight discriminator against interpretation in terms of edges at level-2. They can make loops incapable of return to their roots and wrench an impossible double knot vice versa. If pointwise housing and fractalized nests are eliminated, somewhat easier computation involving edges of the gold quadrille now precludes plausible locality. This preclusion cannot exclude lifetime modifiers because Peters placements do appear flocked.

Only highly localized representations could catch the single trajectory apart from edges and rightly put it in an unflocked basket, the F-complemented pigeonhole. Flesh flies with doubly chained carcass cavity-genetics preview a swarm control tool in starch gradient unit-/fractal genes. Asymptotically equivalent family trees of carriers cannot generate non-overlapping original species. They cannot crash nest security, coat-paint controls. In p generation, swarms rise exhaustively to the (Fibonacci, for example) sentence in (Fibonacci, l grammar after containment at F5. Massive evidence supporting this very real appearance process has been deepened and floated as an answer to Origins.

10.1. Common Misconceptions

Both professionals and the general public may have heard of the Fibonacci sequence recently due to various discoveries, inventions, and artistic creations. However, its historical roots go back a long way and are surprisingly diverse. Several trees in nature exhibit the fibonacci numbers, and mathematical entities can be defined or discovered using the Fibonacci sequence. Many have a Fibonacci flavor to their definition or even courses of conduct, but few may directly involve Fibonacci (Avagyan, 2010). There are many Fibonacci objects and activities. Since the term Fibonacci is attached to a wide range of manifestations including both functional and nonfunctional entities, the term is used in a broad sense in this paper.

Many of the objects that may be referred describe rules of action or processes rather than usual mathematical quantities. Typewriters and cameras can be cited as examples of mechanical objects whose principles and operations are closely associated or derived from the Fibonacci concept, but whose industrial and artistic utilization may not maintain a connection with Fibonacci mathematics (Gorriz, 2023). In a narrower sense, emphasis is placed on Fibonacci numbers, their relative vertices and series, Generalized Fibonacci numbers and related Fibonacci graphs, as well as their Hamiltonians and paths, which are ideal for mathematical, functional, and artistic renditions. So, there are many facets or vertices concerning the mathematical view of Fibonacci objects. To explore their innumerable manifestations, an alternative viewpoint is needed to probe into the mathematical realm of Fibonacci objects.

The textbook definition usually goes like this. The Fibonacci numbers are a sequence of integers, with the first two terms defined as 0 and 1. Each subsequent term is defined as the sum of the two preceding terms in the Fibonacci sequence, for example, the first124 Fibonacci numbers: F(0), F(1), F(2), F(3) = 60, F(4) = 3, F(5) = 5, …, F(123) = 591286729879. They can be represented in many other ways. The sum of the first n Fibonacci numbers is equal to the (n+2) number minus 1. The sum of the squares of the first n Fibonacci numbers is the multiple of the nth and (n+1)th Fibonacci numbers, and so on. The Fibonacci numbers obey numerous mathematical formulas and relations: the negation formula, the addition (subtraction) formula, the fundamental identity, the sine-cosine representation, and multiple-angle formulas. Every 3rd Fibonacci number, starting with F(0), is an even number. Every F(k) with $k \geq 3$ is a multiple of F(4). Every real positive number n can be represented by sum of Fibonacci numbers of the type=F(k). The main relation between the Fibonacci sequence and the golden ratio is that the ratio of two successive terms of the Fibonacci sequence approaches the golden ratio as you look further into the Fibonacci sequence.

10.2. Debunking Myths

Many of them relate to beauty in architecture and nature, emphasizing aesthetics as a universal motive behind their inclusion by artists and architects (Gorriz, 2023). The question here is whether the importance of Fibonacci numbers in mathematics is fundamental to our comprehension of mathematics or aesthetics. Are these triangles relevant to analytical mathematics? Are Fibonacci polynomials, matrices, or sequences intrinsic to mathematics? Could they relate to new results? This will attempt to answer these questions according to the opinions of mathematicians and art historians, emphasizing mathematical accuracy over beauty. The text focused on Fibonaccis polynomials and numbers less known among the public. In dealing with them, although not often commented, very loose equations and estimates on art will lead to manipulations with some contradictions or deceptive conclusions. This can take place either in the analytical field or on the numerical sequences. Considered polynomials or matrices, obtained manipulations should be much clearer. The Golden Section, generated through an irrational number, leads to clear results. On the contrary, the numerical results on Fibonacci numbers leave many questions behind, such as why symmetry on even indices or the Halley's theorem? With these variables, infinite bounds on the time scale could set aside the other dimensions of the example for a better focus and deeper understanding, while Fibonacci-sequence-outlooks are just numeric and give a poorer comprehension. Counting is invisible and encoding relies on reduced signatures. The materials needed for Fibonacci coding are quite different from the desired outcomes. An alternative code, which produces a relative sequence so stable with regard to both size and time scale could be proposed. It allows the rewrite of Fibonacci numbers into binary form with no alternative or representation resistance (Avagyan, 2010).

11. FIBONACCI AND THE GOLDEN RATIO

The Fibonacci numbers introduce a way of thinking about whole numbers as a sequence instead of a list. If they are introduced in this way, it is easy to see how they generate the other mathematical ideas: recursion, limits, the notion of ratio, etc. The Fibonacci numbers are introduced by a simple algorithm: Starting with the two whole numbers zero and one, each new number is the sum of the previous two. This sequence will create a simple yet powerful tool for many applications and fundamental discoveries in mathematics, science and technology. The idea was then applied to create new Fibonacci ratios and Fibonacci geometry. Several non-linear differential equations inspired by concyclic Fibonacci numbers have also been proposed with a successful application to Chaotic Laser Systems. In this context, the Fibonacci ratios were applied to design robust Fibonacci neural networks. These networks can learn and memorize the patterns entering in big data, and they have a robust tolerance even for 90 % of missing data (F. Haight, 2015). The Fibonacci series provides better efficiency and accuracy in such areas as financial predictions, cancer rates and patterns of computer failures. Fibonacci numbers and the golden ratio are found in nearly all domains of Science. The Fibonacci sequence and its mathematical and scientific consequences, discoveries and applications were extensively reviewed (Pletser, 2017). This review presented several examples of Fibonacci numbers and the golden ratio in biology, physics, astrophysics, chemistry and technology. There are many other unexplored questions and examples to develop new areas of research, thus creating more new discoveries, applications and insights. The beauty and ubiquity of Fibonacci numbers and the golden ratio will continue to prompt more new θ-discoveries in science and nature.

11.1. Mathematical Relationship

Between 01 and 02, a star is shown moving through a space of one dimension, with position within that space measuring its state. A pair of numbers is then considered: 0, the additive identity, and 1, the multiplicative identity. Assigning a "representation" to each number yields a second dimension, producing the motion of the movement of 0 and 1 and repeated transformations to explore the state of the movement. A third number, 2, is introduced into the consideration, yielding a third dimension in the representation of numbers. This method of number generation gives meaning to prior digits; biblical, sacred texts, and mathematics are expounded upon in light of this paradigm from the perspective that all is to know about mathematics was told or written by the Numbers.

The Fibonacci sequence and associated numbers are examined from this perspective. The Fibonacci sequence is defined as an infinite series of numbers wherein each number after the first two is generated by summing the previous two numbers. The first two numbers of the sequence are specified as 0 and 1, with subsequent identity values labeled 0 through ∞. The mathematical expression for Fibonacci numbers is shown using the mathematical symbols F_{n} and n, such that $0 \leq n \leq \infty$; with {F(n) values} indicated as 0,1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,a,b,...,0. The Fibonacci stroboscope indicates that the Fibonacci sequence converges to the number φ, along with the "on-finition" of the exponential number e. Graphical representations illustrate gains and losses of Fibonacci numbers connecting to the art of dancing cortexes, the number φ, and Fibonacci's one of the colors of nature.

The "evolution" of Fibonacci numbers is expounded upon, including identity numbers, and the gains and losses of Fibonacci numbers are linked to the observation of f-mozik numbers and Fibonacci stroboscopes, indicating their conversion into surface numbers of g-3D "evolution" based on the γ-component to an "ascended" number in Nature. The in-fractions generated by nature through g-3D evolution in the evolution of Fibonacci numbers are interpreted as non-foliable.

11.2. Applications in Design

The Fibonacci Numbers are widely applied in fine arts, architecture, decoration, poster design, desktop publishing, and graphic design. The elements are organized based on a modification of Fibonacci Theory, instead of the Fibonacci Sequence, therefore the natural structure is excluded to some extent. Besides these applications, the Fibonacci numbers also have a wide range of applications in astronomy, biology, taxonomy, medicine, botany, international security, electronics, optics, architecture, military, photography, sculpture, music, painting, astronomy, aquarium, time, landscaping, textile art, and game design (Avagyan, 2010).

Fibonacci integers in 3D base. The 3D Fibonacci integers, especially the units and the axes, have wider application in animation and modeling. It is also applied in dazzling assembly in matrix format. Due to the limitation of mathematical tools and labors, many mathematical theories are impractically used. The 3D Fibonacci integers and 3D plane embedded Fibonacci integers break through the limitation. It is well known that Fibonacci ratios can produce the Fibonacci spiral or Fibonacci lattice in 2D. The included spirals or lattices in 3D Fibonacci integers are also possible, by formatting the units to possess Fibonacci ratios such as the entry and upper view. In this view, by positioning a particle point based on the 3D Fibonacci integers, it produces the spiral structure. The same mechanism is used to produce the laces of Fibonacci lattice by using 3D surrounded hidden Fibonacci numbers. The width and depth are defined as a fixed number, and a constant delta—0.5 is defined (Pletser, 2017).

The particle follows the path by sitting in the middle of each lattice point with raised motion. It is also possible to add motion in Y-axis at the same time. These actions automatically generate the 3D assembly of Fibonacci ratios. The structure is organized as a mesh of fibers. Each fiber can be tied for decoration or additional functionality. Due to the original ratio, it only produces one optimized position from an infinite number of camera locations in shooting. In 3D animation, if two cameras shifted camera shake is usually required for ambiguity. However, for the Fibonacci integers hold, viewing from other cameras can show an aesthetically pleasing animation, free from camera shake waggler.

12. CULTURAL IMPACT OF FIBONACCI

Fibonacci numbers seem to be scattered all over the universe as a reflection of human interests. Some mathematicians have studied certain properties, relations, or functions of Fibonacci numbers. Others explained the importance of Fibonacci numbers in mathematics, nature, and art. Even though Fibonacci numbers are well studied and popularized, there are still a lot of open problems related to Fibonacci numbers and a lot of research work being done (Avagyan, 2010).

Leonardo Fibonacci (c. 1170-1250) was a mathematician who is often cited in connection with the numbering system now known as Arabic numerals. He is known mostly through his book, Liber Abaci, which appeared in 1202 (Patterson, 2017). As a young man, Fibonacci was taken by his father, a tax collector, to North Africa, where he learned the methods of the “algorists,” or men who did calculations using nine figures and a zero as they are used today. He returned to his home city of Pisa about 1198 and began his lifelong task of introducing to the Western world the “new mathematics” stemming from the Arabic culture. The first step in that task was his famous book, which while not the first in Europe on the subject, was the first to be widely known and popularly accepted. In his exposition on numbers, Fibonacci set forth the arguments in favor of the new numbers, explaining their inferiority to the old Roman system. He also undertook the arithmetic of the new numbers, including their addition, subtraction, multiplication, and division.

12.1. Literature

It is a well-known fact that Fibonacci numbers can be represented in various forms. Here are a few notable examples of their sum identities: The sum of the first n Fibonacci numbers is equal to the (n+2) number minus 1, meaning their values become so big much fastly. The sum of the squares of the first n Fibonacci numbers, resulting from the different combinations in multiplicative format, is the multiple of the nth and (n+1)th Fibonacci numbers. Fibonacci numbers obey numerous mathematical formulas and relations. The Fibonacci numbers obey the negation formula, the addition formula, and the subtraction formula. Each Fibonacci number's index can be alternatively seen from a regard of Islamic design. Every positive integer and any n-non-negative integer can be expressed iteratively with Fibonacci numbers. The Fibonacci sequence of experiments, including petals of same flowers, arrangements of leaves of tree branches, patterns of pine cones and pineapples, distribution of seeds in sunflowers, etc. Many are such experiments around in natural and symbolic life. The Fibonacci numbers have many properties. For example, every 3rd Fibonacci number is an even number. To illustrate, 2

= F(3), 8 = F(5), 34 =F(7), and . . . . At the same time, the to-be-even index is always odd. Alternatively, every odd-index's Fibonacci number is odd, for example, 1 = F(1), 3 =F(4), 13=F(6), ... Furthermore, every 4th Fibonacci number is a multiple of 3, guaranteed from the periodicity of 2 and 5 to be non-negative. Moreover, every real positive number n can be expressed as a sum of Fibonacci numbers, with one of the Fibonacci numbers used at most once, in order that Fi< n< Fi+1 and Fi=Fj+Fk where j<k and j+k+1=i. This is zeckendorff representation of n. Coefficients of the Fibonacci expansions of n follows the zeckendorff theorem, so since there exists unique zeckendorff representation of n, n=Fj1+Fj2+. . . . It is obvious that n > Fi-1 if n=Fj, and n-Fj< Fi-1. Hence, the coefficients follow. The connection between Fibonacci numbers and bionomial coefficients is also famous (Avagyan, 2010). Mathematics and arts, if A and B be quantities, are in the golden ratio if the ratio between the sum of A and B and the larger one is the same as the ratio between the larger one and the smaller. To be precise though, if with quantity A > B, there involves golden ratio ∅ (∅=1.618 . . .etc) if there is a positive number x (x is adding (or subtracting) quantity A in exhibition) in the relationship, that is, A + B = A/s ~ (x) = ∅/1: 1.61803398. Hence, the Fibonacci sequence is in other names phi number sequence or golden number sequence. Related to this relation, there are many ways of showing how the Fibonacci sequence is in accordance with golden ratio, and vice versa (Hansen, 2023). The main relation is that the further one looks at the terms of the sequence, the ratio of the two successive values of terms of Fibonacci sequence becomes closer and closer to the golden ratio, (fn+1)/(fn)=≅∅. In addition, it deserves notice that fn is a little less than or equal to (1/√ 5) (∅n), and for this reason, fn can be well approximated by(1/√ 5) (∅n).

12.2. Philosophy

To me, mathematical beauty revolves around simple ideas giving rise to interesting entities with visualisations that can be sketched on paper and yet have extraordinary properties and depth. To that end, questions arise about the different ways to describe an object, what deeper properties remain invariant across the different descriptions, and which ideas and observations regarding the object are most fundamental. Mathematics is often described by its practitioners as being beautiful. As an example of something both mathematically beautiful and accessible, a geometric object is described with a low level of complexity in terms of the ideas needed to create them and in their visualisation, but a higher level of complexity in terms of the depth and richness of the structure underlying it. Over the following pages a fascinating object will be explained and explored (Hansen, 2023). According to Fermat's principle of least time or principle of least action in classical mechanics, light travels through an array of mediums and obstacles, rays take the path that requires least time. In the main text of this dissertation Fermat's principle is employed to describe, in a low level of entry, the much more complex subject of Fibonacci tiling. In tilings made with a one by one square and a one by φ rectangular thin piece, quasi-periodic properties emerge, such as delta-surface intensity whose squared magnitude yield peaks and sharp transitions. As this quantisation in terms of the Fibonacci sum only yields numbers through infinite summation, an array meditation for finite sized samples on a bias on square area piece is provided. The underlying principles of the Fibonacci approximant valued rational description are examined (F. Haight, 2015).

13. CURRENT RESEARCH TRENDS

The univocal substitution rule on two letters a1 = 0, a2 = 1 leads to all infinite Fibonacci words F on the alphabet A = {0, 1}. There is nothing complementary or reflexive to a starting character; with time the word becomes almost balanced, giving rise to oscillatory behaviour and quasi-periodicity (Hansen, 2023). The number of letters, where f1 is the empty, f2 = a1, ..., f3 is a pair word, grows according to the Fibonacci series. F has no finite periodicity and is neither a simple codification of a unique geometric model or map of simple iterations. This fractal has self similar clusters on different scales, of what appears to be almost uncountably many distinct components assemblages. A deeper investigation into the substitution drawing rules leads to some counter-intuitive results and a surprising insight into this well studied sequence. In the space of lattice paths the Fibonacci word arises as the limit of the 2-D projection coverage of some 3-D conformal images. In describing the coverings a "hairy" growth mechanism is introduced which demonstrates how this 2-D fractal is generated.

In the last 30 years, the mathematical theory of aperiodic order has developed enormously (Baake et al., 2023). Many new tilings and properties have been discovered. There is no doubt that the best known examples of aperiodic order are the Robinson tilings from the early 1970s. Their connection to the golden ratio makes them very appealing, and this appreciation has only grown over the years. Currently, there is hope that detailed experimental results from quasiperiodic crystals could settle open questions concerning the mathematical existence and uniqueness of these tilings. However, there are other sources of aperiodicity that deserve more attention. Here, one of the first and simplest ones, the well-known Fibonacci chain, is presented. It is based on a substitution of the form that generates the well-known Fibonacci words. This family of shapes has several interesting properties, and many stop times are known. Fibonacci words and 2-regular Fibonacci sequences are generalizations of Fibonacci sequences and yield Fibonacci-like lover edges in ubiquitous cascades.

13.1. Mathematical Advances

Three brilliant mathematicians integrated Fibonacci numbers in their own theories. Jacob Bernoulli is notable for studying these numbers in combinatorial applications and establishing stimulating links with probability theory. In particular, he extended the Fibonacci sequences to three arguments. However, he died without publishing and lost priority of the topic. Nevertheless Sylvester published a celebrated paper treating many remarkable reciprocals of Fibonacci quotients which were independently discovered later by S. W. Woolhawk. Moreover, he introduced many interesting notations with generalization, conjectures, and problems. B. G. Voigt also made some important contributions on Fibonacci numbers from heptagonal tilings.

The Fibonacci numbers play a very important role in a variety of situations ranging from the pure mathematics to the sciences. Some mathematicians have devoted their entire lives to studying Fibonacci numbers, and despite their activity, many interesting problems remain unsolved till now. Fibonacci numbers have many properties and applications, and it is impossible to list all these things with references in an article of reasonable length. Since the nature is a mathematical object to model and understand the structures and laws, Fibonacci numbers and their generalizations are convenient and powerful tools in treating many combinatorial, number-theoretic, algebraic and geometric situations. As a result, there should be some interesting, stimulating and natural mathematics. Circular lattice graphs of cities with Fibonacci numbers make a good world and open many fertile prospects.

The Fibonacci sequences 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, … can be defined recursively as F(0) = 0, F(1) = 1, and F(n) = F(n – 1) + F(n – 2) for n ≥ 2 (Avagyan, 2010). The Lucas sequences 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, … can be defined recursively as L(0) = 2, L(1) = 1, and L(n) = L(n – 1) + L(n – 2) for n ≥ 2.

The Fibonacci sequences as natural numbers were studied by many mathematicians even before J. L. L. Fibonacci as observed in the previous section. Moreover, these numbers have been called by various names such as Fibonacci, Lucas, and Lucas–Fibonacci numbers without precise distinctions (Hansen, 2023).

13.2. Interdisciplinary Studies

The studies collectively demonstrate that inter-disciplinal research, which combines mathematics with some other science, has a long-standing history. In anti-Fibonacci research work for instance, the Fibonacci word itself was harnessed to enable the classification of symmetries, couplings and equilibrium points of DNA-like polymers. Not too long before that, Fibonacci words had also been investigated in areas such as biology, thermodynamics, theoretical physics and routing protocols in computer networks. Not only that Fibonacci numbers and series inspired a few research works in quantum walks, they were even made use of to devise its own means of achieving self-cooling via a Quasi-Random Hamiltonian. This tells that Fibonacci series is indeed a beautiful mathematical notion that transcends the borders of various disciplines and holds captivating features in virtually every analytic approach (Hansen, 2023). A paradigm shift is thus called for, aiming to view thing from this distinctive perspective. Indeed, Fibonacci series shares astounding intuitiveness and beauty that is overlooked but is well worth its regard and there is reason to believe that the works of such kind will proliferate. Mathematics, physics, and plants were born and flourished together, and there will be much more to be discovered along this avenue, of which contributions would be expected from researchers having a solid background in both mathematics and botany. For the fervent pursuit of Fibonacci series with embedment of spirit and devoutness, Fibonacci will reward beyond the measure of merit (Marciak-Kozlowska & Kozlowski, 2016).

14. FUTURE DIRECTIONS

So, among its artifacts, there still exists an open offer to us all: to visualize and design, as the Fibonacci sequence grows without bound and its many digital templates pass wider into the deeper recesses of the still mysterious binary universe. For example, observe the close relatives of the Fibonacci numbers, related by half-step recursions, often intended with the more general Pell numbers… The n-th Fibonacci number can be defined as the sum of the two previous ones, as illustrated in the first picture. The squares grow faster than the Fibonacci sequence numbers. It can barely be expected to see any squares along its journey… But the surprises unfold relentlessly with its so-called square-Fibonacci sequence grow larger and larger… The scenes exhibited here clearly reveal two well-separated periodicities but to the vivacious guesses, yet to be verified the intent of the important mathematical relationships, if there is any. Beyond all the randomness called forth by bio-acoustic, bio-electric language, the hope is triumph for early thinkers of mathematical diagrams riches. However, the well-known Fibonacci word 0, 1, to be precise graphic displayed here, has to stop at Turing's IMPOSTER, without better understanding and uncovering its real hidden digital messages encoded in the living language. Although, the simplest Fibonacci tiling, better known as the well-studied L-gominoes, in fact with noticeable beauty re-design starts here, among the taken but abandoned proposals. Gate structures and their properties need attention. The dedicated computer cookies kept up with all sorts of inquiries. More intriguing tilings tagged with the words involving some branches and efforts across the spectrum already existing… but never ever tried out…

14.1. Emerging Applications

The Fibonacci sequence has fascinated people for many years and still finds a wide variety of applications in nature, mathematics, sciences, technology, and art. The modern era of Fibonacci numbers began with almost instant application in their analysis to the so-called traditional approximation of the continuous spectrum of the harmonic Os1, which was done in 1949 by a then student of M.I. Markov in the Moscow State University. A number of applications in physics, biology, astrophysics, and technologies are mentioned; many of them are highlighted in the volumes. The types of devices using Fibonacci numbers are listed. Some of them have been patented recently (Pletser, 2017). Fibonacci numbers define almost exactly the internal morphology of shells in many different mollusk species, the form of branching in many plant species, and the ellipses of a great number of galaxies in different forms. There is speculation that turtles egg clutches are constructed according to Fibonacci spirals. The height of the Galilean telescope is indeed Fibonacci. The heights of many buildings of the same species obey Fibonacci numbers. Successive Fibonacci numbers define Fibonacci filters that are used in rapid temporal Fourier transform analysis and in elimination of non-claimed frequencies from complex telecommunication signals. Fibonacci numbers define buffers providing guaranteed access to densely populated telecommunication busses; this is done by assigning Fibonacci words to addressed messages or cells. On the base of Fibonacci numbers, universal hashing functions are constructed. These functions are used in Microsoft Windows NT. Fibonacci number-based keys provide fast encrypting-decoding of data and access to various systems (Avagyan, 2010).

14.2. Potential Discoveries

Many discoveries can be made from the Fibonacci sequence. For example, it is known that many numbers have a property such that if written in the Fibonacci base representation, the number can be represented in such a way that no two adjacent numbers have a 1. One asks whether this property can be generalized to multiple zeros and/or ones in the base representation.

Another aspect to check could be if the Fibonacci base numbers correspond to a unique graph structure that can be analyzed further. After calculating the Fibonacci word, many underlying fractals can be found that are reminiscent of different structure of fractals. This structure can be compared and contrasted with other fractals and spiral graphs.

Yet another aspect is numerical simulations of applying different transformations to the Fibonacci word, as such techniques will reveal new time and frequency characteristics. New filtering techniques based on the Fibonacci word can be developed, but they need to be detected or devised first, whether in discrete or continuous basis. Fibonacci numbers offer themselves as an alphabet (as do Fibonacci words) to improve numerical algorithms.

Finally, there is the point of applying the concept of adjacent Fibonacci numbers or adjacent Fibonacci words to any other words such as Cantor ternary words or Plain language. Should this self-affinity property exist, many underlying patterns could be exposed (Avagyan, 2010).

15. CONCLUSION

The Fibonacci sequence was discovered as early as 850 AD by the Indian mathematician Pingala. In Europe, in the twelfth century, it was first introduced to the Western world by Leonardo of Pisa. The necessary background of the work of the Fibonacci title. Fibonacci had concerns of monetary exchange with his friend, noting a number of coins of differing weights. He returned an example to show how to exchange coins of different weights in the arithmetic operations. It is assumed that it is done in general. Such consideration leads to the Fibonacci number and the analysis of its source. There were many research activities on the Fibonacci number, most of which are numerical consideration.

It is also well known that many natural phenomena show the regularity of the Fibonacci number. For example, the arrangement of specific leaves in plants, the profile of natural waves, the number of spirals found in sunflower heads, and the number of spirals in pine cones are the Fibonacci numbers. In the universal expansion, the distance between clusters also becomes the Fibonacci numbers. Thus the Fibonacci number appears in many fields. Mathematicians also considered the geometric interpretation of the Fibonacci number. The unit squares with the Fibonacci length on the coordinate plane are drawn, and the constructed figure is called the Fibonacci spiral. It is known that the Fibonacci spiral is close to the growth curve of natural phenomena. Fibonacci was the first to create the modulo series. Though it appears very simple, there are still undiscovered questions. The Fibonacci sequence appears not only in monetary problems but also in geometric series. It can be defined using a point sequence and a substitution style language. The golden angle comes again from the simple property of the Fibonacci sequence.

Prior to his work, the Fibonacci sequence using the bees had been simplified in this homophone of the Arabic language. After that, posed a simplification regarding the factors of the Fibonacci number. This is still open, awaiting investigative advances. The Fibonacci sequence enjoys many more beautiful questions. A Fibonacci number is the sum of the other Fibonacci numbers. There are employee arrangements regarded as recursive structure and substitution drawing on Fibonacci. For example, considering a pigeon coop problem, it is natural to draw a pigeon probability drawing based on a square structure. It was found by trial and error that the Fibonacci numbers were the vertices of each polygon drawn.

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